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Report on the PhD-Thesis
The Cauchy problem for difference equations in lattice cones and generating functions for its solutions
submitted by Sreelatha Chandragiri

The thesis submitted is devoted to solution of a Cauchy problem for difference equations in lattice cones with the help of generating functions.

Difference equations constitute an important class of equations that involve the differences between successive values of a function of a discrete (integer valued) variable. They are often used for description of recurrence relations and have direct applications to Probability Theory and Combinatorics. They can also be seen as discritized analogues of ordinary or partial differential equations.

The theory of one-dimensional difference equations is nowadays a classical branch of Analysis, see e.g. Refs. [1, 15, 19, 38, 42, 43]. Similarly to the case of one-dimensional ordinary differential equations of the n-th order, the space of solutions of a one-dimensional difference equation is finite dimensional. A powerful method to solve difference equations is the method of generating functions (also widely used in discrete probability theory) that can be related to the Laplace method in the theory of ordinary differential equations.

Multi-dimensional difference equations are much more complicated and bear a resemblance to partial differential equations, where — as we know — there is no unified theory. In the linear case, however, one can use the method of fundamental solutions (Green functions) and also the method of multidimensional generating functions to derive a solution. It should be taken into account that the multidimensional setting requires a different approach to boundary conditions.

The thesis consists of an Introduction and three Chapters.

In Chapter 1, the Cauchy problem for a multidimensional inhomogeneous difference equation with constant coefficients in a lattice cone is solved by means of the method of generating functions. In Theorem 1.1, an explicit formula for the generating function of the solution of the Cauchy problem is given. In Theorem 1.3, the formula for the solution is determined in terms of the fundamental solution. The results of Theorem 1.3 are illustrated in Section 1.3 by an application to a lattice path problem.

Chapter 2 is devoted to the analysis of the so-called vector partition functions (with weight φ). In the particular case $\varphi \equiv 1$, the partition function gives the number of representations of a vector belonging to a lattice cone as a linear combination of certain prescribed vectors. Theorem 2.2 establishes a relation between a vector partition function and the generating series for the weight φ. In Theorem 2.5, a very interesting multidimensional generalization of the so-called Chaundy–Bullard identity is obtained. The usefulness of this identity is illustrated later by an application to a path counting problem.

In Chapter 3, the author studies difference equations in a two-dimensional pointed lattice cone spanned by some prescribed vectors. In Theorem 3.1, a general N-dimensional relation for generating functions is derived and a two-dimensional case is discussed in detail. Eventually, these relations are applied to the calculation of the number of the so-called Dyck, Schröder, and Motzkin paths, namely the paths of certain types that stay on or above the main diagonal on a two-dimensional lattice.

The thesis by Ms Chandragiri is devoted to solution of multidimensional lattice Cauchy problems. This topic is important and interesting on its own and has applications, e.g. in Combinatorics and Probability Theory. Multidimensional lattice Cauchy problems are not well studied in general and this thesis fills a certain gap in the theory providing solutions for non-homogeneous linear difference equations in terms of generating functions and fundamental solutions. The methods developed in this thesis generalize the methods known in the one-dimensional case and recall methods used in the theory of linear ordinary and partial differential equations. The validity of the methods is justified by the new results that are consistent with their one-dimensional counterparts as well as by interesting examples. The results of this thesis have been published as journal articles (Refs. [58–61]) and presented at Russian and international conferences (Refs. [62–68]).

The thesis is well written although it contains a number of typos that I list at the end of this report. The core of the thesis is Chapters 1, 2 and 3. They contain the proofs and examples and are formulated in a self-consistent manner and a mathematically precise language. The Summary of the Thesis (pp. 7–17) looks a bit redundant since it is mainly a repetition of some parts of the subsequent chapters. Moreover, the text on p. 36 is a repetition of pp. 31–32; Example on pp. 40–41 coincides with the text on pp. 13–14. A lot of mathematical objects (e.g., $\mathbb{Z}^N = \mathbb{Z} \times \cdots \times \mathbb{Z}$) is repeatedly introduced many times across the text. The thesis could have benefited from an Index of Symbols and an Index of Terminology. I also would have appreciated more examples and a better historical exposition of the subject.

Summing up, in this thesis Ms Sreelatha Chandragiri presented a good piece of original mathematical research work. The results of the thesis are novel, and the arguments are diverse, mathematically rigorous and precise. Ms Chandragiri proved her good command of theories and methods from the fields of difference equations, calculus, and combinatorics.

I believe that this thesis satisfies the criteria of the PhD Regulations of the Siberian Federal University.

With best regards,

Ilya Pavlyukevich

Technical remarks, comments and typos.

p.4 l.-4: "formulae" instead of "a formulae";

p.5: it is not clear if the results were presented by the author herself;

p.7 l.1 and Eq. (0.3): both $\mathbb{Z}_{\geq}$ and $\mathbb{Z}_{\geqslant}$ are used; $\mathbb{Z}^N_{\geqslant}$ was not defined;

p.7 l.2: and Eq. (0.3): it has to be said that $\mathbb{Z}^N = \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{N \text{ times}}$ and that n, N are natural numbers;

p.7 l.5: e^j is a column vector, so $e^j = (0, \dots, 1, \dots, 0)^T$;

p.7 l.6: and everywhere in the text: insert a comma after $j = 1, \dots, N$; the same applies to all similar formulae;

p.7 l.3: $N = 2$, not $n = 2$;

p.8 l.9: it should be $\delta^{-\alpha^j} = (\delta_1^{-\alpha_1^j}, \dots, \delta_n^{-\alpha_n^j})^T$;

p.9 Theorem 1.1 and everywhere in the text: $F(z)$ and $f(x)$ usually denote the values of the functions F and f at z and x respectively; it is better to write F and f or $F(\cdot)$ and $f(\cdot)$;

p.9 l.-8: "satisfies the differential equation";

p.9 l.-3: $\operatorname{supp} \mathcal{P} = \{x \in \mathbb{Z}^n : \mathcal{P}(x) \neq 0\}$;

p.10, Example: again, $\alpha^1 = (1, 0)^T$ etc are vectors; inconsistent notation: α^i and α_i, $i = 1, 2, 3$, are used simultaneously;

p.10 l.-2: and everywhere later: write $\min$ not min;

p.12 l.6: please remove ":" after "by";

p.12 Eq. (0.9): please split the sentence. Start a new sentence "If for a given function h it is ...";

p.13 Theorem 2.5: $|x|$ has to be introduced somewhere;

p.13 l.-4: "which lies"

p.18: the notation has to be explained better: δ_j, δ^{ω}, $\delta_j^{\omega_j}$ are used here;

p.23 l.4: "satisfies the differential equation";

p.24 l.6: "in terms";

p.24 l.8: "Newton polytope" not "Newtone polytope";

p.25 l.-3: please write "from Lemma 1.2"

p.27 Example A: see my remark to p. 10, Example;

p.29 Example B: please write: "Let $\alpha^1 = (2, -1)^T$, $\alpha^2 = (-1, -2)^T$ be column vectors, and let K denote the cone spanned by these vectors...";

p.31, l.-1: what is K_A?

p.32: what is h? Is $P_A(\lambda, \varphi)$ the same as $P_A(x, h)$?

p.33 l.1: the first sentence is incomplete;

p.35 l.-2: please write: "We let K denote the cone spanned by...";

p.35, Proof, and everywhere in the text: please write "the left hand side";

p.49 l.-9: please write "Now regard f as a function on...";

p.49 l.-5: please write "so partial fractions give";

p.50 l.-4: please write "this method exploits";

p.55 l.-11: the sentence "The method..." is not clear;