СИБИРСКИЙ ФЕДЕРАЛЬНЫЙ УНИВЕРСИТЕТ
SIBERIAN FEDERAL UNIVERSITY

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THE CAUCHY PROBLEM FOR DIFFERENCE EQUATIONS IN LATTICE CONES AND GENERATING FUNCTIONS FOR ITS SOLUTIONS

SREELATHA CHANDRAGIRI

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Contents

INTRODUCTION

Difference equations arise in various areas of mathematics, and the theory of linear difference equations in the one-dimensional case is well developed ([1], [7], [19], [21], [22], [38], [42], [43]). In particular, it is known that its space of solutions is finite-dimensional. Constant coefficient difference equations in combination with generating functions are a powerful method used in enumerative combinatorial analysis ([6], [49], [50], [55]) and discrete dynamic systems ([15]).

For multidimensional difference equations, significant difficulties arise primarily from the fact that their space of solutions is infinite dimensional. One way to overcome these difficulties is to impose additional conditions ("initial", "boundary") on their solutions to ensure existence and uniqueness. Selection of additional conditions depends both on the form of the difference equation and on specific problems that require their solutions.

Among the fundamental papers of the theory of multidimensional difference equations is [8], which proves the existence and uniqueness of their solutions. In this paper, the authors used multidimensional difference equations and their generating functions to solve selected classical problems in enumerative combinatorial analysis, including the problem of counting lattice paths in a two-dimensional integer lattice.

Different formulations of the Cauchy problem for multidimensional difference equations were used in [2], [3], [32], [33], [36], [39], [44], [45].

A solution to the Cauchy problem for a multidimensional homogeneous linear difference equation with constant coefficients using the notion of fundamental solution was obtained in [28], [29]. A sufficient solvability condition for Cauchy problems for a polynomial difference operator with constant coefficients was obtained in [35]. The result that if the generating function of the Cauchy data of a homogeneous Cauchy problem lies in one of the classes of Stanley's hierarchy then the generating function of the solution belongs to the same class was proved in [30], [46], [47], [35], [37]. In [57] the Cauchy problem is studied for a multidimensional difference equation in a class of functions defined at the integer points of a rational cone. An easy-to-check condition on the coefficients of the characteristic polynomial of the equation sufficient for solvability of the problem was given. A multidimensional analog of the condition ensuring stability of the Cauchy problem was stated using the notion of amoeba of an algebraic hypersurface.

The numerical stability of multilayer finite difference schemes was studied in [5], [51], [52], [53], [54], using methods of the theory of amoebas of algebraic hypersurfaces. A necessary condition for the stability of a Cauchy problem for a multilayer scheme was given and it was shown that it is not sufficient. Therefore, the authors formulated and proved a sufficient condition for the stability.

In [34], the solvability of the Cauchy problem was shown to be equivalent to the existence of a monomial basis in the quotient ring of the polynomial ring by the ideal generated by the characteristic polynomial.

For one-dimensional equations with constant coefficients the stability of solutions was investigated in the framework of the theory of discrete dynamical systems and was proved to be determined by the roots of the characteristic polynomial, namely: they all lie in the unit disk. Two easily verified sufficient conditions on the coefficients of a difference operator which guarantee the correctness of a Cauchy problem were given [4].

The goal. The goal of our research is to consider the Cauchy problem for a multidimensional difference equation connected with a lattice path problem and obtain a formulae by which the generating function of its solution is expressed in terms of generating functions of the Cauchy data and a solution to the Cauchy problem is expressed through its fundamental solution and Cauchy data; to give an analogue of the Chaundy-Bullard identity for vector partition functions; to derive generating functions of solutions to restricted lattice path problems by using difference equations with non-constant coefficients and methods of diagonal series.

The main results:

- we obtained formulae in which the generating function of the solution to the Cauchy problem is expressed in terms of generating functions of the Cauchy data and a solution to the Cauchy problem is expressed through its fundamental solution and Cauchy data;
- we obtained a Chaundy-Bullard identity for vector partition functions;
- we obtained the identity for the generating functions (series), based on which we derived generating functions of solutions to restricted lattice path problems.

Methods. We use methods of multidimensional complex analysis, including the theory of multidimensional power series and amoebas of algebraic hypersurfaces, and the theory of generating functions (series).

Seminars and Conferences. The results of the work were presented at:

1. «Generating Function for the number of paths on the Multidimensional integer lattice», the City Seminar «Algebraic geometry and multidimensional residues», Krasnoyarsk, Russia, March 2017;
2. «Generating Functions for the Number of Paths on Multidimensional Integer Lattice», the 23rd International Conference on Difference Equations and Applications (ICDEA 2017), Timisoara, Romania, July 2017;
3. «Generating Functions for Lattice Paths and Vector Partitions», the Conference and Workshop «Algebraic Geometry, Complex Analysis and Computer Algebra», Koryazhma, Russia, August 2017;

4. «Generating Functions for Lattice paths and Vector partitions», the City Seminar «Algebraic geometry and multidimensional residues», Krasnoyarsk, Russia, January 2018;

5. «On an Identity of Chaundy and Bullard for Vector Partition Functions», the 24th International Conference on Difference Equations and Applications (ICDEA 2018), Dresden, Germany, May 2018;

6. «Vector Partition Functions based on Chaundy and Bullard Identity», International Student Conference «Prospect Svobodny 2018», Krasnoyarsk, Russia, April 2018;

7. «On Generating Functions for Lattice Paths», ITNOU, 2018;

8. «Generating Functions for Dyck paths, Schröder paths, and Motzkin paths», the City Seminar «Algebraic geometry and multidimensional residues», Krasnoyarsk, Russia, January 2019;

9. «Generating Functions for Some Lattice Paths», International Student Conference «Prospect Svobodny 2019», Krasnoyarsk, Russia, April 2019;

10. «On Fundamental Solutions to Difference Equations in Lattice Cones». International Saratov Winter School «Modern Problems of the Theory of Functions and their Applications», Saratov, Russia, January 2020.

Author's publications. The main results of the research were published in three articles ([58], [59], [60]), in journals indexing by Scopus, and in seven conference proceedings ([62], [63], [64], [65], [66], [67], [68]).

The thesis comprises the introduction, three chapters, the conclusion, and a list of references, and it has 71 pages.

Summary. Let $\mathbb{Z}$ be the set of integers numbers, $\mathbb{Z}_{\geq}$ is the set of non-negative integers and $\mathbb{Z}^N = \mathbb{Z} \times \cdots \times \mathbb{Z}$. We consider the mapping $A : \mathbb{Z}^N \to \mathbb{Z}^n$ defined by a matrix

$$A = \begin{pmatrix} \alpha_1^1 & \ldots & \alpha_1^N \\ \ldots & \ldots & \ldots \\ \alpha_n^1 & \ldots & \alpha_n^N \end{pmatrix}_{n \times N},$$

whose elements are $\alpha_i^j \in \mathbb{Z}, i = 1, \ldots, n, j = 1, \ldots, N$. Denote by $\alpha^j = Ae^j, e^j = (0, \ldots, 1, \ldots, 0), j = 1, \ldots, N$ the columns of matrix A and consider the difference equation

$$f(\lambda) = \sum_{j=1}^{N} c_j f(\lambda - \alpha^j),\ \lambda \in \mathbb{Z}^n. \tag{0.1}$$

In generalized lattice path problems we can consider the difference equation of the form

$$\varphi(x) = c_1\varphi(x - e^1) + \cdots + c_N\varphi(x - e^N),\ x \in \mathbb{Z}^N, \tag{0.2}$$

which is called the basic recurrence relation, because its solutions are binomial coefficients for $n = 2$ and $c_1 = c_2 = 1$.

We can directly verify that if $\varphi(x)$ is a solution to the basic recurrence relation (0.2), then the function

$$f(\lambda) = \sum_{\substack{x: Ax = \lambda \\ x \in \mathbb{Z}^N_{\geqslant}}} \varphi(x) \tag{0.3}$$

is a solution to difference equation (0.1) provided that the sum in (0.3) is defined correctly.

In chapter 1 we consider a variant of the Cauchy problem for a multidimensional difference equation of the form (0.1) with constant coefficients, which connected with a lattice path problem in enumerative combinatorial analysis. We obtained a formula in which the generating function of the solution to the Cauchy problem is expressed in terms of generating functions of the Cauchy data and a formula

expressing the solution to the Cauchy problem through its fundamental solution and Cauchy data.

We describe the space of solutions to difference equations of the form (0.1) in the case when solutions are sought in the pointed cone $K = \langle \alpha^1, \ldots, \alpha^N \rangle$ spanned by vectors $\alpha^j = Ae^j, j = 1, \ldots, N$.

On complex valued functions $f : \mathbb{Z}^n \to \mathbb{C}$ we define the shift operator δ_j as follows:

$$\delta_j f(x) = f(x + e^j), j = 1, \ldots, n,$$

and $\delta^{-\alpha^j} = \delta_1^{-\alpha_1^j}, \ldots, \delta_n^{-\alpha_n^j}, j = 1, \ldots, N$.

For points $u, v \in K$ a partial order relation $\underset{K}{\geqslant}$ is defined as follows: $u \underset{K}{\geqslant} v \Leftrightarrow u - v \in K$. We also denote $u \underset{K}{\not\geqslant} v \Leftrightarrow u - v \notin K$. We assume that the cone K is *pointed*, which means it does not contain any line or, equivalently, lies in an open half-space of $\mathbb{R}^n$.

We consider a finite set of integer points $\{\alpha^1, \ldots, \alpha^N\} \subset K$, in which there exists a point m such that for all $\alpha^j, j = 1, \ldots, N$ the condition $\alpha^j \underset{K}{\leqslant} m$ holds.

Find a function $f : K \to \mathbb{C}$, satisfying the difference equation

$$\sum_{j=0}^{N} c_j f(x - \alpha^j) = g(x), \quad x \underset{K}{\geqslant} m, \tag{0.4}$$

and which coincides with the given function $\varphi(x)$ on the set $X_0 = \{x \in K : x \underset{K}{\not\geqslant} m\}$:

$$f(x) = \varphi(x), \quad x \in X_0. \tag{0.5}$$

The characteristic polynomial for (0.4) is a Laurent polynomial (since it may have terms of negative degree) $P(z) = \sum_{j=0}^{N} c_j z^{-\alpha^j}$.

Equation (0.4) with initial data (0.5) is used to describe a major class of problems in enumerative combinatorial analysis such as lattice path problems (the Dyck, Motzkin, Schröder and generalized lattice paths, see [8], [59], [58]).

The fact that the cone K is pointed allows us to use the method of generating functions. This involves defining for any $\mu \in K$ the element in the ring $\mathbb{C}_K[[z]]$ of (formal) power series

$$F_\mu(z) = \sum_{x \underset{K}{\geqslant} \mu} f(x) z^x.$$

We also define $F(z) = F_0(z)$.

Using the method of generating functions, we will derive a formula which expresses the generating function $F(z)$ in terms of the characteristic polynomial for (0.4) and generating functions for the Cauchy data.

Theorem 1.1. *The generating function $F(z)$ of a solution $f(x)$ to difference equation (0.4) with initial data (0.5) is representable as*

$$F(z) = \frac{1}{P(z^{-1})} \Big(\sum_{j=0}^{N} c_j z^{\alpha^j} \Phi_{m-\alpha^j}(z) + G_m(z) \Big), \tag{0.6}$$

where $P(z^{-1}) = P(z_1^{-1}, \dots, z_n^{-1})$, $\Phi_{m-\alpha^j}(z) = F(z) - F_{m-\alpha^j}(z)$ and $G_m(z) = \sum\limits_{x \underset{K}{\geqslant} m} g(x) z^x$.

A function $\mathcal{P} : \mathbb{Z}^n \to \mathbb{C}$ is called *a fundamental solution* to the Cauchy problem (0.4)–(0.5) if it satisfies to the difference equation

$$\sum_{j=0}^{N} c_j \mathcal{P}(x - \alpha^j) = \delta_0(x), \quad x \in \mathbb{Z}^n, \tag{0.7}$$

where $\delta_0(x)$ is the Kronecker symbol:

$$\delta_0(x) = \begin{cases} 0, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0. \end{cases}$$

The support of the function $\mathcal{P}(x)$ is a set

$$\operatorname{supp} \mathcal{P}(x) = \{x \in \mathbb{Z}^n : \mathcal{P}(x) \neq 0\}.$$

The concept of the fundamental solution $\mathcal{P}(x)$ to (0.4)–(0.5) yields a formula expressing $f(x)$ in terms of Cauchy data $\varphi(x)$ and $g(x)$.

Theorem 1.3. *A solution to the difference equation (0.4) with initial data (0.5) is given as follows:*

$$f(x) = \sum_{0 \underset{K}{\leqslant} y \underset{K}{\leqslant} x} \mathcal{P}(x-y)\tau(y),$$

where $\tau(y) = \begin{cases} \sum\limits_{j=0}^{N} c_j \varphi(y - \alpha^j), \ \textit{if}\ y \underset{K}{\not\geqslant} m; \\ g(y), \ \textit{if}\ y \underset{K}{\geqslant} m. \end{cases}$

Example. We consider a set with three steps $A = \{\alpha^1 = (1,0), \alpha^2 = (0,1), \alpha^3 = (1,1)\}$ and let $f(x_1, x_2)$ denote the number of paths from the origin to $(x_1, x_2) \in \mathbb{Z}^2$ using steps from the set A. The cone K is spanned by the vectors from A and $m = \alpha_1 + \alpha_2$, since $\alpha_3 = \alpha_1 + \alpha_2$.

We consider the two dimensional difference equation

$$f(x_1, x_2) - f(x_1 - 1, x_2) - f(x_1, x_2 - 1) - f(x_1 - 1, x_2 - 1) = 0, \qquad (0.8)$$

and its characteristic polynomial $P(z_1, z_2) = 1 - z_1^{-1} - z_2^{-1} - z_1^{-1} z_2^{-1}$.

By Theorem 1.3 a solution to difference equation (0.8) is

$$f(x_1, x_2) = \sum_{0 \underset{K}{\leqslant} y \underset{K}{\leqslant} x} \mathcal{P}(x_1 - y_1, x_2 - y_2)\tau(y_1, y_2),$$

where $\tau(y_1, y_2) = \varphi(y_1, y_2) - \varphi(y_1 - 1, y_2) - \varphi(y_1, y_2 - 1) - \varphi(y_1 - 1, y_2 - 1)$ if $(y_1, y_2) \not\geqslant (1,1)$ and $\tau(y_1, y_2) = 0$ otherwise.

Consequently, Lemma 1.2 gives the fundamental solution to (0.8)

$$\mathcal{P}(x_1, x_2) = \sum_{t=0}^{min(x_1, x_2)} \frac{(x_1 + x_2 - t)!}{(x_1 - t)!(x_2 - t)!t!} = \sum_{\substack{k_1 + k_3 = x_1 \\ k_2 + k_3 = x_2 \\ k_1, k_2, k_3 \geqslant 0}} \frac{(k_1 + k_2 + k_3)!}{k_1! k_2! k_3!} = P_A(x; h).$$

Finally, we obtain the solution for difference equation (0.8) with initial data function $f(x_1,x_2)=\varphi(x_1,x_2), (x_1,x_2)\ngeq(1,1)$ as follows

$$f(x_1,x_2)=\mathcal{P}(x_1,x_2)\varphi(0,0)+\sum_{y_1=1}^{x_1}\mathcal{P}(x_1-y_1,x_2)(\varphi(y_1,0)-\varphi(y_1-1,0))+$$
$$+\sum_{y_2=1}^{x_2}\mathcal{P}(x_1,x_2-y_2)(\varphi(0,y_2)-\varphi(0,y_2-1)).$$

In the case of lattice paths, $\varphi(y_1,0)-\varphi(y_1-1,0)=0$ for $y_1\geqslant 1,\ \varphi(0,y_2)-\varphi(0,y_2-1)=0$ for $y_2\geqslant 1,$ and $\varphi(0,0)=1$, we obtain

$$f(x_1,x_2)=\mathcal{P}(x_1,x_2).$$

In chapter 2 a sufficient condition for correctness of solution to equation (0.1) of the form (0.3) is given, which consists in the fact that $f(\lambda)$ is a vector partition function associated with function $\varphi(x)$.

For the function $\varphi:\mathbb{Z}^N_{\geq}\to\mathbb{C}$, for $\lambda\in K_A$ we define a vector partition function associated with φ, as follows

$$P_A(\lambda;\varphi)=\sum_{\substack{x:Ax=\lambda\\ \lambda\in K}}\varphi(x).$$

If $\varphi(x)=1$, then this is a classical vector partition function, i.e. number of representations of the vector λ, belonging to the lattice cone K, in the form of a linear combination of given vectors $\alpha^j, j=1,\dots,N$. Another classical interpretation of the function $P_A(\lambda;\varphi)$ is the number of solutions to the Diophantine system of linear equations $Ax=\lambda$ (see [55]).

The function $P_A(\lambda;\varphi)$ will also be called *the vector partition function with weight* $\varphi(x)$.

We define a generalized vector partition function and derive an identity for generating series of such functions associated with solutions of basic recurrence relation of combinatorial analysis. As a consequence we obtain the generating function of the number of generalized lattice paths and a new version of the Chaundy-Bullard identity for the vector partition function.

For further discussion and formulation of the theorem 2.2 and theorem 2.5 we introduce some notations. Let $V = \{J\}$ be a set of all ordered sets $J = (j_1, j_2, \ldots, j_k)$, $1 \leqslant j_1 < \ldots < j_k \leqslant N, k = 0, 1, 2, \ldots, N$ and $\#J = k$ be a number of elements in the set J. Let π_j be the projection operator along the j-th coordinate axis in $\mathbb{R}^n$, i.e. $\pi_j x = (x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_N)$, and define its adjoint action on the function $\varphi(x) : \mathbb{Z}^N \to \mathbb{C}$ by: $\pi_j \varphi(x) = \varphi(\pi_j x), j = 1, \ldots, N$. Let $\mathbb{C}[[\xi]]$ be the ring of formal power series in the variable $\xi = (\xi_1, \ldots, \xi_N)$ and define the operator $\pi_j : \mathbb{C}[[\xi]] \to \mathbb{C}[[\xi]]$ for $j = 1, \ldots, N$ on the generating series $\Phi(\xi) = \sum\limits_{x \in \mathbb{Z}^N} \varphi(x)\xi^x$ as follows

$$\pi_j \Phi(\xi) = \sum_{x \in \mathbb{Z}^N} \varphi(\pi_j x)\xi^{\pi_j x} = \Phi(\xi_1, \ldots, \xi_{j-1}, 0, \xi_{j+1}, \ldots, \xi_N).$$

Furthermore, for $J \in V$ let $\pi_J = \pi_{j_1} \circ \cdots \circ \pi_{j_k}$ be a composition of operators $\pi_{j_1}, \ldots, \pi_{j_k}$ and $\pi_\emptyset = 1$ is the identity operator.

For $c = (c_1, \ldots, c_N) \in \mathbb{C}^N$ we denote the operator $Q(\delta) = \delta_1 \cdot \ldots \cdot \delta_N - c_1\delta_2 \cdot \ldots \cdot \delta_N - \cdots - c_N\delta_1 \cdot \ldots \cdot \delta_{N-1}$ and $z^A = (z^{\alpha^1}, \ldots, z^{\alpha^N})$.

A basic identity in the theory of summation of functions is the following, practically tautological, relation:

$$\varphi(x) - \varphi(0) = \sum_{k=1}^{x} (\varphi(k) - \varphi(k-1)), x \in \mathbb{N} \tag{0.9}$$

if for a given function $h(x)$ it is possible to find a function $\varphi(x)$, such that $\varphi(x) - \varphi(x-1) = h(x)$, then (2.2) becomes

$$\varphi(k)\Big|_0^x = \sum_{k=1}^{x} h(k), \tag{0.10}$$

which is the discrete analogue of the Newton-Leibniz formula, and the function $\varphi(x)$ is a discrete primitive for the function $h(x)$. For the generating function $\Phi(\xi) = \sum\limits_{x=0}^{\infty} \varphi(x)\xi^x, \xi \in \mathbb{C}$, identity (2.2) is equivalent to

$$\Phi(\xi) - \frac{\Phi(0)}{1-\xi} = \frac{1}{1-\xi} \sum_{x=1}^{\infty} (\varphi(x) - \varphi(x-1))\, \xi^x \tag{0.11}$$

or

$$(1-\xi)\Phi(\xi)-\Phi(0)=\sum_{x=1}^{\infty}\left(\varphi(x)-\varphi(x-1)\right)\xi^{x}. \qquad (0.12)$$

Now we formulate a multidimensional analogue of identity (0.12).

Theorem 2.2. *Let $P_A(\lambda;\varphi)$ be the vector partition function, associated with a function $\varphi:\mathbb{Z}^N_{\geqslant}\to\mathbb{C}$, and $\Phi(\xi)$ be the generating series for $\varphi(x)$. Then*

$$\sum_{J\in V}(-1)^{\#J}\pi_J\left[(1-\langle c,\xi\rangle)\Phi(\xi)\right]\Big|_{\xi=z^A}=\sum_{\lambda\in K\cap\mathbb{Z}^N}P_A(\lambda;Q(\delta)\varphi)z^{\lambda}. \qquad (0.13)$$

For $j=1,\dots,N$ we denote $\Delta_j=\Delta\setminus\{\alpha^j\}$ and $A_j=[\alpha^1,\dots,\alpha^{j-1},\alpha^{j+1},\dots,\alpha^N]$, then $K_j=\{\nu\in\mathbb{Z}^n:\nu=y_1\alpha^1+\dots[j]\dots+y_N\alpha^N, y\in\mathbb{Z}^N_{\geqslant}\}$; $c^x=c_1^{x_1}\cdots c_N^{x_N}$. Note that each cone $K_j\subset K$ is also pointed.

We give an analogue of the Chaundy-Bullard identity for vector partition functions.

Theorem 2.5. *If $c_1+c_2+\dots+c_N=1$ and $\varphi_j(x)=\frac{|x|!}{x!}c^{x+e^j}$, then for any $\mu=(\mu_1,\dots,\mu_N)\in\mathbb{Z}^N$ the identity*

$$\sum_{j=1}^{N}\sum_{\nu\in K_j}P_{A_j}(\nu)P_A(\mu-\nu;\varphi_j)=P_A(\mu) \qquad (0.14)$$

holds.

The sum on the left side of identity (0.14) is finite since all the cones $K_j, j=1,\dots,N$ are pointed.

Note that for $\alpha^j=e^j, j=1,\dots,N$ we obtain from (0.14) a multidimensional Chaundy-Bullard identity:

$$\sum_{j=1}^{N}\sum_{\substack{0\leqslant\nu\leqslant\mu\\ \nu_j=0}}\frac{(|\mu|-|\nu|)!}{(\mu-\nu)!}c^{\mu-\nu+e^j}\equiv 1,$$

where the double inequality $0\leqslant\nu\leqslant\mu$ means that $0\leqslant\nu_j\leqslant\mu_j$ for all $j=1,\dots,N$.

Example. Let's consider a set of 3 steps: $\Delta = \{\alpha^1 = (1,1), \alpha^2 = (1,0), \alpha^3 = (1,-1)\}$. Then for $\lambda = (4,1)$, $\varphi(x) = 1$ we have $P_A(\lambda;\varphi) = P_A(\lambda)$ is a number of non-negative integer solutions to the system

$$\begin{cases} x_1 + x_2 + x_3 & = 4 \\ x_1 - x_3 & = 1 \end{cases}.$$

It has 2 solutions $(1,3,0)$ and $(2,1,1)$, which also means that we can get from the origin to the point $\lambda = (4,1)$ using 2 different sequences of the steps: $\alpha_1\alpha_2\alpha_2\alpha_2$ and $\alpha_1\alpha_1\alpha_2\alpha_3$.

This allows us to compute the number of paths from the origin to the point λ:

$$f(\lambda) = C_4^1 C_3^3 + C_4^2 C_2^1 C_1^1.$$

For $c_1 + c_2 + c_3 = 1$ we get the Chaundy-Bullard identity

$$\begin{aligned} c_1(12c_1^2c_2c_3 + 4c_1c_2^3 + 3c_1^2c_2 + 3c_1^2c_3 + 3c_1c_2^2 + c_1^2 + 2c_1c_2 + c_1) + \\ + c_2(12c_1^2c_2c_3 + 4c_1c_2^3 + 3c_1^2c_2 + 6c_1c_2c_3 + c_2^3 + 2c_1c_2 + 2c_2c_3 + c_2) + \\ c_3(12c_1^2c_2c_3 + 4c_1c_2^3 + 3c_1^2c_3 + 3c_1c_2^2 + 6c_1c_2c_3 + c_2^3 + \\ 2c_1c_2 + 2c_1c_3 + c_2^2 + 2c_2c_3 + c_1 + c_2 + c_3 + 1) = 2. \end{aligned}$$

Chapter 3 deals with equations of the form (0.1) when the pointed cone $K = \langle \alpha^1, \dots, \alpha^N \rangle \subset \mathbb{Z}^n$ is spanned by n linearly independent vectors $\{\alpha^1, \dots, \alpha^n\}$ (we can choose them without loss of generality) from a system of vectors $\{\alpha^1, \dots, \alpha^N\}$. Any $\lambda \in K$ can be represented in a unique way in the form $\lambda = \nu_1\alpha^1 + \dots + \nu_n\alpha^n$, $\nu_j \in \mathbb{Z}_\geqslant, j = 1, \dots, n$, and the vectors $\alpha^{n+1}, \dots, \alpha^N$ can be represented in the form $\alpha^{n+1} = \beta_1^{n+1}\alpha^1 + \dots + \beta_n^{n+1}\alpha^n, \dots, \alpha^N = \beta_1^N\alpha^1 + \dots + \beta_n^N\alpha^N$, $\beta_j^i \in \mathbb{Z}_\geqslant$. If $g(\nu) = f(\nu_1\alpha^1 + \dots + \nu_n\alpha^n)$, then the difference equation takes the form

$$g(\nu) = c_1 g(\nu - \beta^1) + \dots + c_n g(\nu - \beta^n) + c_{n+1} g(\nu - \beta^{n+1}) + \dots + c_N g(\nu - \beta^N), \quad (0.15)$$

where $\beta^j \in \mathbb{Z}_\geqslant^n, j = 1, \dots, N$.

The method described above from difference equations of the form (0.1) to difference equations of the form (0.15) can be used in the problem of listing lattice paths when moving a point in a two-dimensional integer lattice. It is, in particular, useful for classical problems involving Dyck, Motzkin and Schröder paths.

In the case of arbitrary n, this allows the use of methods developed in [8] for finding generating function for equations of the form (0.15) to obtain formulas for the generating functions of the solution of equation (0.1).

Let $x, m, \alpha \in \mathbb{Z}^N_{\geqslant}, P(z) = \sum\limits_{0 \leqslant \alpha \leqslant m} c_\alpha z^\alpha$ be a polynomial in $z \in \mathbb{C}^N$. The inequality $0 \leqslant \alpha \leqslant m$ means that $0 \leqslant \alpha_j \leqslant m_j$ for all $j = 1, \ldots, N$. We denote $F_\alpha(z) = \sum\limits_{x \geqslant \alpha} f(x) z^x$ and $\Phi_\alpha(z) = F(z) - F_\alpha(z)$, where the inequality $x \ngeqslant \alpha$ means, that for at least one $j_0 \in \{1, \ldots, N\}$ the inequality $x_{j_0} < \alpha_{j_0}$ holds.

Let $P(\delta) = \sum\limits_{0 \leqslant \alpha \leqslant m} c_\alpha \delta^\alpha$ be a polynomial difference operator with constant coefficients.

We first derive a general identity for the generating functions, we note that this theorem is an analogue of identity (0.12) for generating functions. We employ a difference equation with non-constant coefficients and illustrate this theorem by using it to count Dyck, Schröder, Motzkin and generalized lattice paths.

Theorem 3.1. *For any $F(z) \in \mathbb{C}[[z]]$ the identity*

$$P(z)F(z) - \sum_{0 \leqslant \alpha \leqslant m} c_\alpha z^\alpha \Phi_{m-\alpha}(z) = \sum_{x \geqslant m} P(\delta^{-I}) f(x) z^x \tag{0.16}$$

holds, where $I = (1, \ldots, 1)$.

We will present an example connected with well-known lattice paths – Dyck paths (see [8], [27], [40], [18], [13], [12], [55]). We use a linear transformation which reduces the mentioned lattice paths to the lattice path which lie in $\mathbb{Z}^2_{\geqslant}$ in order to use methods for finding generating functions developed [30] and [26]. However, to study lattice paths lying on or over a rational slope, linear difference equations with non-constant coefficients will be used to put restrictions on them.

Example. *Dyck paths* start at the origin and stay on or above the main diagonal $y = x$ (see [8], [40], [41]) using steps $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Let $f(x, y)$ denote the number of paths going from $(0, 0)$ to (x, y). The number of paths $f(x, y)$ satisfies the difference equation

$$f(x,y) - f(x-1,y) - f(x,y-1) = -\delta_0(x-y-1)f(x-1,y), \tag{0.17}$$

where $\delta_0(x) = \begin{cases} 1, & \text{if} \quad x = 0 \\ 0, & \text{if} \quad x \neq 0 \end{cases}$ is a Kronecker symbol, with the initial data:

$$f(x,0) = 0, \;\; x = 1,2,\ldots, \quad f(0,y) = 1, \;\; y = 0,1,2,\ldots. \tag{0.18}$$

Let $F_{11}(t)$ be a diagonal power series of $F(z_1, z_2)$:

$$F_{11}(t) = \sum_{k=1}^{\infty} f(k,k)t^k.$$

Proposition 3.3. *Let $F(z_1, z_2)$ be the generating function of the solution of (0.17). Then the series $F(z_1, z_2)$ satisfy the following functional equation*

$$(1 - z_1 - z_2)F(z_1,z_2) - (1 - z_1)F(z_1,0) - (1 - z_2)F(0,z_2) + F(0,0) = \\ = -z_1 \sum_{k \geqslant 1} f(k,k)(z_1 z_2)^k.$$

If the solution of $f(x, y)$ satisfies the initial conditions (0.18), then we get a diagonal power series

$$F_{11}(t) = \sum_{k=1}^{\infty} f(k,k)t^k = \frac{1 - 2t - \sqrt{1-4t}}{2t} = t + 2t^2 + 5t^3 + 14t^4 + 42t^5 \cdots . \tag{0.19}$$

Other examples of lattice paths are presented in Chapter 3.

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Chapter 1. Difference equations in lattice cones and their fundamental solutions

We consider a variant of the Cauchy problem for a multidimensional difference equation with constant coefficients, which is connected with a lattice path problem in enumerative combinatorial analysis. We obtained a formula in which the generating function of the solution to the Cauchy problem is expressed in terms of generating functions of the Cauchy data and a formula expressing solution to the Cauchy problem through its fundamental solution and Cauchy data.

1.1 Definitions and preliminary results

On complex valued functions $f : \mathbb{Z}^n \to \mathbb{C}$ we define the shift operator δ_j as follows:

$$\delta_j : f(x_1, \ldots, x_j, \ldots, x_n) \mapsto f(x_1, \ldots, x_j + 1, \ldots, x_n)$$

and the polynomial difference operator

$$P(\delta) = \sum_{\omega \in \Omega} c_\omega \delta^\omega,$$

where $\Omega \subset \mathbb{Z}^n$ is a finite set of points of an n-dimensional lattice, $\delta^\omega = \delta_1^{\omega_1} \cdot \ldots \cdot \delta_n^{\omega_n}$ and $c_\omega \in \mathbb{C}$ are the coefficients of the difference operator.

We consider the difference equation

$$P(\delta) f(x) = g(x), \quad x \in X, \tag{1.1}$$

where $f(x)$ is an unknown function, and $g(x)$ is a function defined on some set $X \subset \mathbb{Z}^n$. Also choose a set $X_0 \subset \mathbb{Z}^n$, the points of which will be called initial (boundary) points.

In the general situation we have to solve *the Cauchy problem*: find a function $f(x)$, satisfying equation (1.1) and coinciding with a given function $\varphi(x)$ of initial data on the set X_0 :

$$f(x) = \varphi(x), \quad x \in X_0. \tag{1.2}$$

The function $g(x)$ in the right-hand side of (1.1) and the initial data function $\varphi(x)$ in (1.2) is called *the Cauchy data* of problem (1.1)–(1.2).

Existence and uniqueness of problem (1.1)-(1.2) (solvability of the Cauchy problem) depends on all the objects involved in its formulation: the difference operator $P(\delta)$, the set X on which the right part of the equation is given, and the set X_0 on which the initial data $\varphi(x)$ is defined.

In the one-dimensional case two variants of the Cauchy problems are usually considered:

(i) $X = \{x \in \mathbb{Z} : x \geqslant 0\}$ is the set of non-negative integers, $P(\delta) = \sum\limits_{\omega=0}^{m} c_\omega \delta^\omega$, $X_0 = \{0, 1, \ldots, m-1\}$, $c_m \neq 0$,

(ii) $X = \{x \in \mathbb{Z} : x \geqslant m\}$, $P(\delta) = \sum\limits_{\omega=0}^{m} c_\omega \delta^{-\omega}$, $X_0 = \{0, 1, \ldots, m-1\}$, $c_0 \neq 0$.

For example, option (i) is used to describe the solution to equation (1.1) in the theory of discrete dynamic systems (see [15]). Option (ii) is most useful in problems of enumerative combinatorial analysis (see [55]).

In the case of constant coefficients, the z-transformation

$$F(z) = \sum_{x=0}^{\infty} \frac{f(x)}{z^x}$$

is the powerful method to study discrete dynamical systems and generating functions

$$F(z) = \sum_{x=0}^{\infty} f(x) z^x$$

are used for studying problems in enumerative combinatorial analysis.

In the multi-dimensional case, the number of formulations of Cauchy problem (1.1)–(1.2) increases. We discuss some of them.

An analogue of the one-dimensional case (ii), when $X = \mathbb{Z}^n_{\geqslant}$ is the non-negative octant in $\mathbb{Z}^n$, $X_0 = \mathbb{Z}^n_{\geqslant} \setminus X_m$, $0 \in \Omega$, $m_i = \max\{\omega_i : \omega_i \in \Omega, i = 1, \dots, n\}$, $m = (m_1, \dots, m_n)$ and $X_m = \{x \in \mathbb{Z}^n_{\geqslant} : x_i \geqslant m_i, i = 1, \dots, m\}$, is considered in [8], which is devoted to multi-dimensional difference equations with constant coefficients and their use in enumerative combinatorial analysis. Several equivalent conditions, providing solvability of problem (1.1)–(1.2), are given in Theorem 3 in [8]. Particularly, the convex hull $\operatorname{conv}\{\Omega \setminus \{0\}\} \cap \mathbb{R}^n_{\geqslant}$ is not empty.

Various analogues of variant (i) of the Cauchy problem for the multi-dimensional case are constructed as follows. Let $\{\alpha^1, \dots, \alpha^N\}$ be the set of vectors $\alpha^j = (\alpha^j_1, \dots, \alpha^j_n) \in \mathbb{Z}^n, j = 1, \dots, N$, and K is a lattice cone spanned by these vectors

$$K = \{x \in \mathbb{Z}^n : x = \lambda_1 \alpha^1 + \dots + \lambda_N \alpha^N, \lambda_j \in \mathbb{Z}_{\geqslant}, j = 1, \dots, N\}.$$

For points $u, v \in K$ a partial order relation $\underset{K}{\geqslant}$ is defined as follows: $u \underset{K}{\geqslant} v \Leftrightarrow u - v \in K$. We also denote $u \underset{K}{\ngeqslant} v \Leftrightarrow u - v \notin K$. We assume that the cone K is *pointed*, which means it does not contain any line or, equivalently, lies in an open half-space of $\mathbb{R}^n$.

We consider a finite set of integer points $\{\alpha^1, \dots, \alpha^N\} \subset K$, in which there exists a point m such that for all $\alpha^j \in \{\alpha^1, \dots, \alpha^N\}, j = 1, \dots, N$ the condition $\alpha^j \underset{K}{\leqslant} m$ holds.

The solvability of the problem when the cone K is simplicial (which means that every element in it admits a unique expansion in the generators) and the sets $X = K$ and $X_0 = X \setminus (m + K)$, on which Cauchy problem (1.1)–(1.2) is solved, was studied in [2], [4], [29], [30], [35], [34], [51], [52], [53], [54], [57]. Additionally, in these papers, the solutions $f(x)$ to problem (1.1)–(1.2) are given in terms of the Cauchy data and fundamental solution to (1.1)–(1.2) (the Green function). These solutions play an important role in the study of asymptotics of solutions to the Cauchy problem, in particular, to study the stability of the problem and its connection with the properties of the characteristic set

$$\mathcal{V}_P := \{z \in \mathbb{C}^n : P(z) := \sum_{\omega\in\Omega} c_\omega z^\omega \;=\; 0\}$$

of the equation (1.1), where $z^\omega = z_1^{\omega_1} \cdot \ldots \cdot z_N^{\omega_N}$.

A multidimensional analogue of option (ii) for the Cauchy problem(1.1)–(1.2) was not described in [8]. This is apparently due to the fact that in problems of enumerative combinatorial analysis the search for the generating function for the combinatorial object is considered as a full solution to the problem, rather than the study of its asymptotic behavior.

In the next section we formulate a variant of a Cauchy problem for $n > 1$ which combines multidimensional analogs of (i) and (ii) for which the simplicity of the cone K is not required.

1.2 The Cauchy problem and main results

The Cauchy problem is to find a function $f : K \to \mathbb{C}$, satisfying the difference equation

$$\sum_{j=0}^{N} c_j f(x - \alpha^j) = g(x), \quad x \underset{K}{\geqslant} m, \tag{1.3}$$

where $m = \alpha^1 + \cdots + \alpha^N, c_0 = 1, \alpha^0 = (0, \ldots, 0)$ and which coincides with the given function $\varphi(x)$ on the set $X_0 = \{x \in K : x \underset{K}{\ngeqslant} m\}$:

$$f(x) = \varphi(x), \quad x \in X_0. \tag{1.4}$$

The characteristic polynomial for (1.3) is a Laurent polynomial (since it may have terms of negative degree) $P(z) = \sum\limits_{j=0}^{N} c_j z^{-\alpha^j}$.

Equation (1.3) with initial data (1.4) is used to describe a major class of problems in enumerative combinatorial analysis including lattice path problems (the Dyck, Motzkin, Schröder and generalized lattice paths, see [8], [27], [40]).

The fact that the cone K is pointed allows us to use the method of generating functions. This involves defining for any $\mu \in K$ the element in the ring $\mathbb{C}_K[[z]]$ of (formal) power series

$$F_\mu(z) = \sum_{x \underset{K}{\geqslant} \mu} f(x) z^x.$$

We also define $F(z) = F_0(z)$.

Using the method of generating functions, we will derive a formula which expresses the generating function $F(z)$ in terms of the characteristic polynomial for (1.3) and generating functions for the Cauchy data.

Theorem 1.1. *The generating function $F(z)$ of a solution $f(x)$ to difference equation (1.3) with initial data (1.4) is representable as*

$$F(z) = \frac{1}{P(z^{-1})} \Big(\sum_{j=0}^{N} c_j z^{\alpha^j} \Phi_{m-\alpha^j}(z) + G_m(z) \Big), \tag{1.5}$$

where $P(z^{-1}) = P(z_1^{-1}, \ldots, z_n^{-1})$, $\Phi_{m-\alpha^j}(z) = F(z) - F_{m-\alpha^j}(z)$ and $G_m(z) = \sum\limits_{x \underset{K}{\geqslant} m} g(x) z^x$.

Proof. Multiplying each side of (1.3) by z^x and summing over $x \underset{K}{\geqslant} m$ yields

$$\sum_{x \underset{K}{\geqslant} m} f(x) z^x + \sum_{x \underset{K}{\geqslant} m} c_1 f(x - \alpha^1) z^x + \ldots + \sum_{x \underset{K}{\geqslant} m} c_N f(x - \alpha^N) z^x = \sum_{x \underset{K}{\geqslant} m} g(x) z^x,$$

$$F_m(z) + c_1 z^{\alpha^1} \sum_{x \underset{K}{\geqslant} m} f(x - \alpha^1) z^{x - \alpha^1} + \ldots + c_N z^{\alpha^N} \sum_{x \underset{K}{\geqslant} m} f(x - \alpha^N) z^{x - \alpha^N} = G_m(z),$$

$$F_m(z) + c_1 z^{\alpha^1} \sum_{x + \alpha^1 \underset{K}{\geqslant} m} f(x) z^x + \ldots + c_N z^{\alpha^N} \sum_{x + \alpha^N \underset{K}{\geqslant} m} f(x) z^x = G_m(z),$$

$$F_m(z) + c_1 z^{\alpha^1} F_{m-\alpha^1}(z) + \ldots + c_N z^{\alpha^N} F_{m-\alpha^N}(z) = G_m(z),$$

$$(F(z) - \Phi_m(z)) + c_1 z^{\alpha^1} (F(z) - \Phi_{m-\alpha^1}(z)) + \ldots + c_N z^{\alpha^N} (F(z) - \Phi_{m-\alpha^N}(z)) = G_m(z),$$

$$(1 + c_1 z^{\alpha^1} + \ldots + c_N z^{\alpha^N}) F(z) = \Phi_m(z) + c_1 z^{\alpha^1} \Phi_{m-\alpha^1}(z) + \ldots + c_N z^{\alpha^N} \Phi_{m-\alpha^N}(z) + G_m(z).$$

Thus we obtain (1.5), which proves the theorem. $\square$

Remark. *Formulae* (1.5) *was derived in [39] for the Riordan arrays and in [30] for* $K = \mathbb{Z}^N$ *and* $g(x) = 0$.

A function $\mathcal{P} : \mathbb{Z}^n \to \mathbb{C}$ is called *a fundamental solution* to the Cauchy problem (1.3)–(1.4) if it satisfies to the difference equation

$$\sum_{j=0}^{N} c_j \mathcal{P}(x - \alpha^j) = \delta_0(x), \quad x \in \mathbb{Z}^n, \tag{1.6}$$

where $\delta_0(x)$ is the Kronecker symbol:

$$\delta_0(x) = \begin{cases} 0, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0. \end{cases}$$

The support of the function $\mathcal{P}(x)$ is a set

$$\operatorname{supp} \mathcal{P}(x) = \{x \in \mathbb{Z}^n : \mathcal{P}(x) \neq 0\}.$$

Lemma 1.2. If $\mathcal{P}(x)$ is the fundamental solution to Cauchy problem (1.3)–(1.4) and $\operatorname{supp} \mathcal{P} \subset K$, where K is a pointed cone, then

$$P(z^{-I}) \cdot \sum_{x \in \mathbb{Z}^n} \mathcal{P}(x) z^x = 1. \tag{1.7}$$

Proof. The product

$$\sum_{j=0}^{N} c_j z^{\alpha^j} \cdot \sum_{x \in \mathbb{Z}^n} \mathcal{P}(x) z^x = \sum_{j=0}^{N} \sum_{x \in \mathbb{Z}^n} c_j \mathcal{P}(x) z^{x+\alpha^j} =$$

$$= \sum_{x \in \mathbb{Z}^n} \sum_{j=0}^{N} c_j \mathcal{P}(x - \alpha^j) z^x = \sum_{x \in \mathbb{Z}^n} \delta_0(x) = 1,$$

which proves the lemma. □

The fundamental solution is

$$\mathcal{P}(x) = \sum_{\substack{A\lambda = x \\ \lambda \in \mathbb{Z}^N_{\geqslant}}} \frac{(-c_1)^{\lambda_1} \cdot \ldots \cdot (-c_N)^{\lambda_N} (\lambda_1 + \cdots + \lambda_N)!}{\lambda_1! \ldots \lambda_N!}, \quad x \underset{K}{\geqslant} 0,$$

and can be obtained by expanding $\frac{1}{P(z^{-1})}$ into the Laurent series as follows:

$$\frac{1}{P(z^{-1})} = \frac{1}{1 - \sum\limits_{j=1}^{N}(-c_j)z^{\alpha^j}} = \sum_{k=0}^{\infty}\left(\sum_{j=1}^{N}(-c_j)z^{\alpha^j}\right)^k =$$

$$= \sum_{\lambda_1+\ldots+\lambda_N \geqslant 0} \frac{(-c_1)^{\lambda_1}\cdot\ldots\cdot(-c_N)^{\lambda_N}(\lambda_1+\cdots+\lambda_N)!}{\lambda_1!\ldots\lambda_N!} z^{\lambda_1\alpha^1+\ldots+\lambda_N\alpha^N} =$$

$$= \sum_{x \underset{K}{\geqslant} 0} \sum_{\substack{A\lambda=x \\ \lambda\in\mathbb{Z}^N_{\geqslant}}} \frac{(-c_1)^{\lambda_1}\cdot\ldots\cdot(-c_N)^{\lambda_N}(\lambda_1+\cdots+\lambda_N)!}{\lambda_1!\ldots\lambda_N!} z^x = \sum_{x \underset{K}{\geqslant} 0} \mathcal{P}(x)z^x.$$

The Laurent series $\sum\limits_{x \underset{K}{\geqslant} 0} \mathcal{P}(x)z^x$ converges in a domain which can be described in term of an amoeba $\mathcal{A}_P$ of the Laurent polynomial $P(z)$. Namely, the logarithmic image of the domain is a complement component of the amoeba $\mathcal{A}_P$ corresponding to the point 0 of the Newtone polytope $\mathcal{N}_P$ (see [17]).

The function $P_A(x;h) = \sum\limits_{\substack{A\lambda=x \\ \lambda\in\mathbb{Z}^N_{\geqslant}}} h(\lambda)$ is called *the vector partition function associated with* $h(\lambda)$. Provided that $h(\lambda) = \frac{(-c)^\lambda|\lambda|!}{\lambda!}$, we get

$$\mathcal{P}(x) = P_A(x;h). \tag{1.8}$$

For $h(\lambda) \equiv 1$ the vector partition function $P_A(x;h) = P_A(x)$ equals the number of non-negative integer solutions to a linear Diophantine equation $A\lambda = x$ (see, for example, [55]):

$$P_A(x) = \sum_{\substack{A\lambda=x \\ \lambda\in\mathbb{Z}^N_{\geqslant}}} 1, \quad x \in \mathbb{Z}^n.$$

For $h(\lambda) = e^{-\langle\lambda,y\rangle}$ properties of the function

$$P_A(y;x) = \sum_{\substack{A\lambda=x \\ \lambda\in\mathbb{Z}^N_{\geqslant}}} e^{-\langle\lambda,y\rangle}, y \in \mathbb{C}^N, \tag{1.9}$$

called *the vector partition function associated with the set of vectors* A, were investigated in [9]. In particular, the authors derived the residue formulas for its generating function and for an analog of the Euler-Maclaurin formula, in which the vector partition functions are represented as the action of the Todd operator on the volume function of a polyhedron. Furthermore, a sum of $e^{-\langle\lambda,y\rangle}$ in integer cones was investigated in [48] in connection to a generalization of the Riemann-Roch theorem. A structure theorem for the vector partition function was presented and polyhedral tools for the efficient computation of such functions was provided in [56].

For $h(\lambda)\equiv 1$ the function $P_A(x;h)$ coincides with the classical vector partition function. For $h(\lambda)=e^{-\langle\lambda,y\rangle}$ we obtain a vector partition function of the form (1.9). If we take $N=2$, $A=(1\ 1)$ and $h(\lambda_1,\lambda_2)=h(\lambda_1)$, then $P_A(x;h)=\sum\limits_{\substack{\lambda_1+\lambda_2=x\\ \lambda_1,\lambda_2\geqslant 0}} h(\lambda_1)=\sum\limits_{\lambda_1=0}^{x} h(\lambda_1)$. Thus, the problem of finding the vector partition function $P_A(x;h)$ is a generalization of the classical summation problem for functions of a discrete argument.

The concept of the fundamental solution $\mathcal{P}(x)$ to (1.3)–(1.4) yields a formula expressing $f(x)$ in terms of Cauchy data $\varphi(x)$ and $g(x)$.

Theorem 1.3. *A solution to the difference equation (1.3) with initial data (1.4) is given as follows:*

$$f(x)=\sum_{0\underset{K}{\leqslant} y\underset{K}{\leqslant} x}\mathcal{P}(x-y)\tau(y),$$

where $\tau(y)=\begin{cases}\sum\limits_{j=0}^{N}c_j\varphi(y-\alpha^j), & \text{if } y\underset{K}{\ngeqslant} m;\\ g(y), & \text{if } y\underset{K}{\geqslant} m.\end{cases}$

Proof. Using expression (1.5) from Theorem 1.1 and expression (1.7) from Lemma yields

$$F(z)=\sum_{x\underset{K}{\geqslant}0}\mathcal{P}(x)z^x\left(\sum_{j=0}^{N}c_jz^{\alpha^j}\Phi_{m-\alpha^j}(z)+G_m(z)\right).$$

Since

$$\sum_{j=0}^{N} c_j z^{\alpha^j} \Phi_{m-\alpha^j}(z) = \sum_{j=0}^{N} c_j z^{\alpha^j} \sum_{\substack{x \underset{K}{\geqslant} 0 \\ x \underset{K}{\ngeqslant} m-\alpha^j}} \varphi(x) z^x =$$

$$= \sum_{j=0}^{N} \sum_{\substack{x \underset{K}{\geqslant} 0 \\ x \underset{K}{\ngeqslant} m-\alpha^j}} c_j \varphi(x) z^{x+\alpha^j} = \sum_{j=0}^{N} \sum_{\substack{y \underset{K}{\geqslant} \alpha^j \\ y \underset{K}{\ngeqslant} m}} c_j \varphi(y-\alpha^j) z^y =$$

$$= \sum_{j=0}^{N} \sum_{\substack{y \underset{K}{\geqslant} 0 \\ y \underset{K}{\ngeqslant} m}} c_j \varphi(y-\alpha^j) z^y = \sum_{\substack{y \underset{K}{\geqslant} 0 \\ y \underset{K}{\ngeqslant} m}} \left(\sum_{j=0}^{N} c_j \varphi(y-\alpha^j) \right) z^y.$$

thus we get

$$F(z) = \sum_{x \underset{K}{\geqslant} 0} \mathcal{P}(x) z^x \sum_{y \underset{K}{\geqslant} 0} \tau(y) z^y,$$

where $\tau(y) = \begin{cases} \sum\limits_{j=0}^{N} c_j \varphi(y-\alpha^j), & \text{if } y \underset{K}{\ngeqslant} m; \\ g(y), & \text{if } y \underset{K}{\geqslant} m. \end{cases}$

Finally, taking into account that $\mathcal{P}(x) = 0$ for $x \underset{K}{\ngeqslant} 0$ we get

$$F(z) = \sum_{x \underset{K}{\geqslant} 0} \left(\sum_{y \underset{K}{\geqslant} 0} \mathcal{P}(x)\tau(y) \right) z^{x+y} = \sum_{x \underset{K}{\geqslant} 0} \left(\sum_{0 \underset{K}{\leqslant} y \underset{K}{\leqslant} x} \mathcal{P}(x-y)\tau(y) \right) z^x.$$

Equating the coefficients of z^x we obtain

$$f(x) = \sum_{0 \underset{K}{\leqslant} y \underset{K}{\leqslant} x} \mathcal{P}(x-y)\tau(y),$$

which proves the theorem. □

1.3 Lattice paths and the Cauchy problem

A lattice path is a finite sequence $p_0, p_1, \ldots, p_L$ of points in $\mathbb{Z}^n$ and its steps are the finite set of lattice vectors $p_k - p_{k-1} \in A = \{\alpha^1, \ldots, \alpha^N\}, k = 1, 2, \ldots, L.$

The common class of lattice paths arises by imposing some conditions on the paths: points $p_k, k = 0, 1, \dots, L$, are distinct (non intersecting paths). In the context of lattice path counting problems the function $f : \mathbb{Z}^N \to \mathbb{Z}_{\geqslant}$ that counts the number $f(x)$ of paths in a specified class for which $p_0 = 0$ is computed (the condition $p_0 = 0$ does not result in a loss of generality). Examples of some well-known lattice paths: Dyck, Motzkin and Schröder paths (for more details see [8], [12], [18], [27]).

It is well-known that the function $f(x)$ satisfies difference equation (1.3) with $c_0 = 1, c_1 = \dots = c_N = -1$ and $g(x) = 0$ (see [8]). Thus $P(\delta) = 1 - \delta^{-\alpha^1} - \dots - \delta^{-\alpha^N}$.

Theorem 1.3 yields a simple formula for the number $f(x)$ of such paths (see also [58]). The following condition for an initial data function $\varphi(x)$ of Cauchy problem (1.3)–(1.4) for the lattice path problem holds:

$$\varphi(x) = \begin{cases} 0, & \text{if } x \underset{K}{\not\geqslant} 0; \\ 1, & \text{if } x = 0; \\ (1 - P(\delta))\varphi(x), & \text{if } x \underset{K}{\geqslant} 0, x \neq 0. \end{cases}$$

Since $\tau(y)$ is equal to 1 only at the origin and vanishes at other points we get $f(x) = \mathcal{P}(x)$. Considering (1.8) we obtain

$$f(x) = P_A(x; h), \text{ where } h(\lambda) = \frac{|\lambda|!}{\lambda!}.$$

Now we will illustrate this idea and give two examples.

Example A.

We consider a set with three steps $A = \{\alpha^1 = (1, 0), \alpha^2 = (0, 1), \alpha^3 = (1, 1)\}$ and let $f(x_1, x_2)$ denote the number of paths from the origin to $(x_1, x_2) \in \mathbb{Z}^2$ using steps from the set A. The cone K is spanned by the vectors from A and $m = \alpha_1 + \alpha_2$, since $\alpha_3 = \alpha_1 + \alpha_2$.

We consider the two dimensional difference equation

$$f(x_1, x_2) - f(x_1 - 1, x_2) - f(x_1, x_2 - 1) - f(x_1 - 1, x_2 - 1) = 0, \qquad (1.10)$$

and its characteristic polynomial $P(z_1, z_2) = 1 - z_1^{-1} - z_2^{-1} - z_1^{-1}z_2^{-1}$.

By Theorem 1.3 a solution to this difference equation is

$$f(x_1, x_2) = \sum_{0 \underset{K}{\leqslant} y \underset{K}{\leqslant} x} \mathcal{P}(x_1 - y_1, x_2 - y_2)\tau(y_1, y_2),$$

where $\tau(y_1, y_2) = \varphi(y_1, y_2) - \varphi(y_1 - 1, y_2) - \varphi(y_1, y_2 - 1) - \varphi(y_1 - 1, y_2 - 1)$ if $(y_1, y_2) \not\geqslant (1, 1)$ and $\tau(y_1, y_2) = 0$ otherwise.

To find the fundamental solution $\mathcal{P}(x_1, x_2)$ we expand $P^{-1}(z_1^{-1}, z_2^{-1})$ as follows

$$\frac{1}{P(z_1^{-1}, z_2^{-1})} = \frac{1}{1 - (z_1 + z_2 + z_1z_2)} = \sum_{k=0}^{\infty}(z_1 + z_2 + z_1z_2)^k =$$

$$= \sum_{k_1,k_2,k_3 \geqslant 0} \frac{(k_1 + k_2 + k_3)!}{k_1!k_2!k_3!} z_1^{k_1} z_2^{k_2} (z_1z_2)^{k_3} = \sum_{x_1,x_2 \geqslant 0} \sum_{t=0}^{min(x_1,x_2)} \frac{(x_1 + x_2 - t)!}{(x_1 - t)!(x_2 - t)!t!} z_1^{x_1} z_2^{x_2}.$$

Consequently, Lemma 1.2 gives

$$\mathcal{P}(x_1, x_2) = \sum_{t=0}^{min(x_1,x_2)} \frac{(x_1 + x_2 - t)!}{(x_1 - t)!(x_2 - t)!t!} = \sum_{\substack{k_1+k_3=x_1 \\ k_2+k_3=x_2 \\ k_1,k_2,k_3 \geqslant 0}} \frac{(k_1 + k_2 + k_3)!}{k_1!k_2!k_3!} = P_A(x; h).$$

Finally, we obtain the solution for difference equation (1.10) with initial data function $f(x_1, x_2) = \varphi(x_1, x_2), (x_1, x_2) \not\geqslant (1, 1)$ as follows

$$f(x_1, x_2) = \mathcal{P}(x_1, x_2)\varphi(0, 0) + \sum_{y_1=1}^{x_1} \mathcal{P}(x_1 - y_1, x_2)(\varphi(y_1, 0) - \varphi(y_1 - 1, 0))+$$

$$+ \sum_{y_2=1}^{x_2} \mathcal{P}(x_1, x_2 - y_2)(\varphi(0, y_2) - \varphi(0, y_2 - 1)).$$

In the case of lattice paths, $\varphi(y_1, 0) - \varphi(y_1 - 1, 0) = 0$ for $y_1 \geqslant 1$, $\varphi(0, y_2) - \varphi(0, y_2 - 1) = 0$ for $y_2 \geqslant 1$, and $\varphi(0, 0) = 1$, we obtain

$$f(x_1, x_2) = \mathcal{P}(x_1, x_2).$$

Example B.

Let $\alpha^1 = (2, -1), \alpha^2 = (-1, 2)$ be a column vectors, we let K denote the cone K spanned by the vectors $K = \langle \alpha^1, \alpha^2 \rangle, m = \alpha^1 + \alpha^2 = (1, 1)$.

We consider the two dimensional difference equation

$$f(x_1, x_2) - f(x_1 - 2, x_2 + 1) - f(x_1 + 1, x_2 - 2) = 0 \qquad (1.11)$$

and its characteristic polynomial $P(z_1, z_2) = 1 - z_1^{-2} z_2 - z_1 z_2^{-2}$.

By Theorem 1.3 a solution to this difference equation is

$$f(x_1, x_2) = \sum_{0 \underset{K}{\leqslant} y \underset{K}{\leqslant} x} \mathcal{P}(x_1 - y_1, x_2 - y_2) \tau(y_1, y_2),$$

where $\tau(y_1, y_2) = \begin{cases} \varphi(y_1, y_2) - \varphi(y_1 - 2, y_2 + 1) - \varphi(y_1 + 1, y_2 - 2), \\ \qquad \text{if } (y_1, y_2) \underset{K}{\ngeqslant} (1, 1) \\ 0, \text{ if } (y_1, y_2) \underset{K}{\geqslant} (1, 1). \end{cases}$

To find a fundamental solution $\mathcal{P}(x_1, x_2)$ we expand the function $P^{-1}(z_1^{-1}, z_2^{-1})$ into a series:

$$\frac{1}{1 - z_1^2 z_2^{-1} - z_1^{-1} z_2^2} = \sum_{k=0}^{\infty} (z_1^2 z_2^{-1} + z_1^{-1} z_2^2)^k = \sum_{k_1 + k_2 \geqslant 0} \frac{(k_1 + k_2)!}{k_1! k_2!} (z_1^2 z_2^{-1})^{k_1} (z_1^{-1} z_2^2)^{k_2} =$$

$$= \sum_{k_1 + k_2 \geqslant 0} \frac{(k_1 + k_2)!}{k_1! k_2!} z_1^{2k_1 - k_2} z_2^{-k_1 + 2k_2} = \sum_{(x_1, x_2) \underset{K}{\geqslant} 0} \frac{(x_1 + x_2)!}{\left(\frac{2x_1 + x_2}{3}\right)! \left(\frac{x_1 + 2x_2}{3}\right)!} z_1^{x_1} z_2^{x_2}.$$

Consequently, by Lemma 1.2

$$\mathcal{P}(x_1, x_2) = \frac{(x_1 + x_2)!}{\left(\frac{2x_1 + x_2}{3}\right)! \left(\frac{x_1 + 2x_2}{3}\right)!}.$$

Finally, we have the solution for difference equation (1.11) with arbitrary initial data

$$f(x_1, x_2) = \varphi(x_1, x_2), (x_1, x_2) \underset{K}{\ngeqslant} (1, 1)$$

$$f(x_1,x_2) = \mathcal{P}(x_1,x_2)\varphi(0,0)+\sum_{t=1}^{x_1}\mathcal{P}(x_1-2t,x_2+t)(\varphi(2t,-t)-\varphi(2t-2,-t+1))+$$
$$+\sum_{t=1}^{x_2}\mathcal{P}(x_1+t,x_2-2t)(\varphi(-t,2t)-\varphi(-t+1,2t-2)).$$

In the case of lattice paths, $\varphi(2t,-t)-\varphi(2t-2,-t+1)=0$ for $t\geqslant 1,$ $\varphi(-t,2t)-\varphi(-t+1,2t-2)=0$ for $t\geqslant 1,$ and $\varphi(0,0)=1$, we obtain

$$f(x_1,x_2)=\mathcal{P}(x_1,x_2).$$

Chapter 2. Generating functions for vector partition functions and a basic recurrence relation

We define a generalized vector partition function and derive an identity for generating series of such functions associated with solutions of the basic recurrence relation of combinatorial analysis. As a consequence we obtain the generating function of the number of generalized lattice paths and a new version of the Chaundy-Bullard identity for the vector partition function.

2.1 Vector partition function and its properties

Denote $\mathbb{Z}^n = \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{n \text{ times}}$, where $\mathbb{Z}$ is the set of integers, $\mathbb{Z}_{\geq}$ is the set of non-negative integers. Consider the set of vectors $\alpha^j = (\alpha_1^j, \ldots, \alpha_n^j)$ with integers coefficients, $j = 1, \ldots, N$ and the matrix $A = \left(\alpha_i^j\right)_{N \times n}$, whose columns are the coordinates of the vectors α^j. We assume that the cone

$$K = \{\lambda \in \mathbb{R}^n : \lambda = x_1 \alpha^1 + \cdots + x_N \alpha^N, x_j \in \mathbb{R}_{\geq}, j = 1, \ldots, N\},$$

is *pointed*, which means it does not contain any line or equivalently lies in an open half-space of $\mathbb{R}^n$ and generated by the vectors α^j. In the future we will denote by the same letter K a lattice cone whose points λ are linear combinations of vectors α^j with non-negative integer coefficients:

$$K = \{\lambda \in \mathbb{Z}^n : \lambda = x_1 \alpha^1 + \cdots + x_N \alpha^N, x_i \in \mathbb{Z}_{\geq}, j = 1, \ldots, N\}.$$

In general case every point $\lambda \in K$ can be represented in the form $\lambda = Ax$ for several different points $x \in \mathbb{Z}^N_{\geq}$.

For the function $\varphi : \mathbb{Z}^N_{\geq} \to \mathbb{C}$, for $\lambda \in K_A$ we define **a vector partition**

function associated with φ, as follows

$$P_A(\lambda;\varphi) = \sum_{\substack{x:Ax=\lambda\\ \lambda\in K}} \varphi(x).$$

If $\varphi(x) = 1$, then this is a classical vector partition function, i.e. number of representations of the vector λ, belonging to the lattice cone in the form of a linear combination of given vectors α^j. Another classical interpretation of the function $P_A(\lambda;\varphi)$ is the number of solutions of the Diophantine system of linear equations $Ax = \lambda$ (see [55]).

Note. The function $P_A(\lambda;\varphi)$ will also be called the vector partition function with weight $\varphi(x)$.

For $h(\lambda) \equiv 1$ the vector partition function $P_A(x;h) = P_A(x)$ is a number of non-negative integer solutions to a linear Diophantine equation $A\lambda = x$ (see, for example, [55]):

$$P_A(x) = \sum_{\substack{A\lambda=x\\ \lambda\in\mathbb{Z}^N_{\geqslant}}} 1, \quad x\in\mathbb{Z}^n.$$

For $h(\lambda) = e^{-\langle\lambda,y\rangle}$ properties of the function

$$P_A(y;x) = \sum_{\substack{A\lambda=x\\ \lambda\in\mathbb{Z}^N_{\geqslant}}} e^{-\langle\lambda,y\rangle}, y\in\mathbb{C}^N, \tag{2.1}$$

called *the vector partition function associated with the set of vectors A*, were investigated in [9]. In particular, they derive the residue formulas for its generating function and an analog of the Euler-Maclaurin formula, in which the vector partition functions are represented as the action of the Todd operator on the volume function of a polyhedron. Furthermore, a sum of $e^{-\langle\lambda,y\rangle}$ in integer cones was investigated in [48] in connection to generalization of the Riemann-Roch theorem. A structure theorem for the vector partition function was presented and polyhedral tools for the efficient computation of such functions was provided in [56].

Find the relationship between the generating function of the weight function $\varphi : \mathbb{Z}^N \to \mathbb{C}$ and the generating function of the associated vector partition function $P_A(\lambda;\varphi)$ associated with it.

The support $\operatorname{supp}\varphi$ of the generating function (series) $\Phi(\xi) = \sum_{x\in\mathbb{Z}^N} \varphi(x)\xi^x$ is the set $\operatorname{supp}\varphi = \{x \in \mathbb{Z}^N : \varphi(x) \neq 0\}$. Note that if the support of the weight function $\operatorname{supp}\varphi \subset \mathbb{Z}^N_{\geq}$, then for the corresponding feature vector of the partition $\operatorname{supp} P_A(\lambda;\varphi) \subset K$.

Proposition 2.1. *If* $\Phi(\xi) = \sum_{x\in\mathbb{Z}^N} \varphi(x)\xi^x$ *is the generating function for the function* $\varphi : \mathbb{Z}^N \to \mathbb{C}$ *and* $\operatorname{supp} P_A(\lambda;\varphi) \subset K$*, then for the generating series* $R_A(z) = \sum_{\lambda\in K} P_A(\lambda;\varphi)z^\lambda$ *functions of the vector partition, the formula* $R_A(\lambda;\varphi) = \Phi(z^A)$*, holds, where* $z^A = (z^{\alpha^1}, \dots, z^{\alpha^N})$.

Proof. Because

$$\Phi(\xi) = \sum_{x\in\mathbb{Z}^N} \varphi(x)\xi^x = \sum_{x\in\mathbb{Z}^N_{\geq}} \varphi(x)\xi^x,$$

after replacing $\xi = z^A$, taking into account the fact that $\operatorname{supp} P_A(\lambda;\varphi) \subset K$ and $A : \mathbb{Z}^N_{\geq} \to K$ we get

$$\Phi(z^A) = \sum_{\lambda\in K} \left(\sum_{\substack{x:Ax=\lambda \\ x\in\mathbb{Z}^N_{\geq}}} \varphi(x) \right) z^\lambda = \sum_{\lambda\in K} P_A(\lambda;\varphi)z^\lambda = R_A(z;\varphi).$$

The condition that the cone K is pointed, ensures the correctness of the definition of the vector partition function $P_A(\lambda;\varphi)$, since the number of solutions of the system of Diophantine equations $Ax = \lambda$ is finite. □

Example 1. Let $\varphi(x) = \begin{cases} 1, & x \in \mathbb{Z}^N_{\geq}; \\ 0, & x \notin \mathbb{Z}^N_{\geq}; \end{cases}$ then

$$\Phi(\xi) = \sum_{x\in\mathbb{Z}^N_{\geq}} \xi^x = \frac{1}{(1-\xi_1)\cdots(1-\xi_N)}.$$

It follows from the proposition that the generating function $R_A(z;1)$ for the vector function the partition $P_A(\lambda;1)$ is equal to

$$R_A(z;1) = \frac{1}{(1-z^{\alpha^1})\cdots(1-z^{\alpha^N})}.$$

Example 2. For the vector partition function $P_A(\lambda;\varphi)$, where $\varphi(x) = e^{-\langle y,x\rangle}$ we find

$$\Phi(\xi) = \sum_{x\in\mathbb{Z}^N_{\geq}} e^{-\langle y,x\rangle}\xi^x = \sum_{x\in\mathbb{Z}^N_{\geq}} (e^{-y_1}\xi_1)^{x_1}\cdots(e^{-y_N}\xi_N)^{x_N} = \frac{1}{(1-e^{-y_1\xi_1})\cdots(1-e^{-y_N\xi_N})}.$$

The last equality is true if $|e^{-y_j\xi_j}| < 1, j = 1,\dots,N$. The generating function $R_A(z;e^{-\langle y,x\rangle})$ for the vector partition function has the form

$$R_A(z;e^{-\langle y,x\rangle}) = \frac{1}{(1-e^{-\langle y,z^{\alpha^1}\rangle})\cdots(1-e^{-\langle y,z^{\alpha^N}\rangle})}$$

for z satisfying the conditions $|e^{-\langle y,z^{\alpha^j}\rangle}| < 1, j = 1,\dots,N$.

Example 3. In the classical problem of summing a function $\varphi(x)$ we have to find a function $s(\lambda)$ such that $s(\lambda) = \sum\limits_{x=0}^{\lambda}\varphi(x)$. We note that $s(x)$ is a special case of the vector partition function $s(\lambda) = \sum\limits_{x_1+x_2=\lambda}\varphi(x_1)$ for $\alpha^1 = \alpha^2 = (1)$.

Proposition (2.1) in this case means that if $\Phi(\xi_1) = \sum\limits_{x_1=0}^{\infty}\varphi(x_1)\xi_1^{x_1}$ is a generating function for $\varphi(x_1)$, then the generating function for the sum $s(\lambda)$ is equal to $\frac{\Phi(z)}{1-z}$.

In the general case summation of the function $\varphi(x) = \varphi(x_1,\dots,x_N)$ over integer points rational polyhedron can be reduced to finding the vector partition function with weight $\varphi(x)$.

We define a generalized vector partition function and derive an identity for generating series of such functions associated with solutions of basic recurrence relation of combinatorial analysis. As a consequence we obtain the generating function of the number of generalized lattice paths and a new version of the Chaundy-Bullard identity for the vector partition function.

2.2 Statement of the main results

A basic identity in the theory of summation of functions is the following, practically tautological, relation:

$$\varphi(x) - \varphi(0) = \sum_{k=1}^{x} (\varphi(k) - \varphi(k-1)), x \in \mathbb{N}. \tag{2.2}$$

If for a given function $h(x)$ it is possible to find a function $\varphi(x)$, such that $\varphi(x) - \varphi(x-1) = h(x)$, then (2.2) becomes

$$\varphi(k)\Big|_0^x = \sum_{k=1}^{x} h(k), \tag{2.3}$$

which is the discrete analogue of the Newton-Leibniz formula, and the function $\varphi(x)$ is a discrete primitive for the function $h(x)$. For the generating function $\Phi(\xi) = \sum\limits_{x=0}^{\infty} \varphi(x)\xi^x, \xi \in \mathbb{C}$, identity (2.2) is equivalent to

$$\Phi(\xi) - \frac{\Phi(0)}{1-\xi} = \frac{1}{1-\xi} \sum_{x=1}^{\infty} (\varphi(x) - \varphi(x-1)) \xi^x \tag{2.4}$$

or

$$(1-\xi)\Phi(\xi) - \Phi(0) = \sum_{x=1}^{\infty} (\varphi(x) - \varphi(x-1)) \xi^x. \tag{2.5}$$

We define a generalized vector partition function associated with a set of lattice vectors and with a complex-valued function of integer arguments. We give a multidimensional analogue of identity (2.5) in Theorem 2.2 and use it to investigate some properties of generalized lattice paths (Propositions 2.3 and 2.4) and prove a version of the Chaundy-Bullard identity for the generalized vector partition function (Theorem 2.5, see [10]).

Let $\mathbb{Z}$ be the set of integers, $\mathbb{Z}^n = \mathbb{Z} \times \cdots \times \mathbb{Z}$ and $\Delta = \{\alpha^1, \alpha^2, \ldots, \alpha^N\} \subset \mathbb{Z}^n$ be a finite set of column vectors. Let $\mathbb{R}^n_{\geqslant}$ be a subset of $\mathbb{R}^n$ with non-negative coordinates. We let K denote the cone K spanned by the vectors in Δ:

$$K = \{\lambda \in \mathbb{R}^n : \lambda = x_1\alpha^1 + \cdots + x_N\alpha^N, x \in \mathbb{R}^N_{\geqslant}\}.$$

We assume that cone K is *pointed*, which means it does not contain any line or equivalently lies in an open half-space of $\mathbb{R}^n$. We let $A = [\alpha^1, \ldots, \alpha^N]$ denote $(n \times N)$-matrix composed of the column vectors in Δ.

The vector partition function $P_A(\lambda)$ of $\lambda \in \mathbb{Z}^n$ is (see, for example, [55]) the number of non-negative integer solutions to a linear Diophantine equation $\alpha^1 x_1 + \ldots + \alpha^N x_N = \lambda$:

$$P_A(\lambda) = \sum_{\substack{x: Ax=\lambda \\ x \in \mathbb{Z}^N_{\geqslant}}} 1, \quad \lambda \in \mathbb{Z}^n. \tag{2.6}$$

Geometrically the function $P_A(\lambda)$ equals the number of representations of the vector λ as a linear combination of the vectors in Δ with non-negative integer coefficients.

In [9] properties of the function

$$P_A(y; \lambda) = \sum_{\substack{x: Ax=\lambda \\ x \in \mathbb{Z}^N_{\geqslant}}} e^{-\langle x, y \rangle}, y \in \mathbb{C}^N, \tag{2.7}$$

called *the vector partition function associated with the set of vectors* Δ, are investigated. In particular, they derive the residue formulas for its generating function and an analog of the Euler-Maclaurin formula, in which the vector partition functions are represented as the action of the Todd operator on the volume function of a polyhedron.

Furthermore, a sum of $e^{\langle x, y \rangle}$ in integer cones was investigated in [48] in connection to generalization of the Riemann-Roch theorem. A structure theorem for the vector partition function was presented and polyhedral tools for the efficient computation of such functions was provided in [56].

For an arbitrary function of integer arguments $\varphi : \mathbb{Z}^N_{\geqslant} \to \mathbb{C}$ we define a function

$$P_A(\lambda; \varphi) = \sum_{\substack{x: Ax=\lambda \\ x \in \mathbb{Z}^N_{\geqslant}}} \varphi(x), \lambda \in \mathbb{Z}^n$$

which we call *the vector partition function associated with* $\varphi(x)$.

For $\varphi(x) \equiv 1$ then the function $P_A(\lambda; \varphi)$ coincides with the classical function of the vector partition (2.6). For $\varphi(x) = e^{-\langle x,y \rangle}$ we obtain a vector partition function of the form (2.7). If we take $N = 2, A = (1\ 1)$ and $\varphi(x_1, x_2) = h(x_1)$, then $P_A(\lambda; \varphi) = \sum\limits_{\substack{x_1+x_2=\lambda \\ x_1,x_2 \geqslant 0}} h(x_1) = \sum\limits_{x_1=0}^{\lambda} h(x_1)$. Thus, the problem of finding the vector partition function $P_A(\lambda; \varphi)$ is a generalization of the classical summation's problem of functions of a discrete argument.

For further discussion and formulation of the main result we introduce some notations. Let $V = \{J\}$ be a set of all ordered sets $J = (j_1, j_2, \dots, j_k)$, $1 \leqslant j_1 < \dots < j_k \leqslant N, k = 0, 1, 2, \dots, N$ and $\#J = k$ be a number of elements in the set J. Let π_j be the projection operator along the j-th coordinate axis in $\mathbb{R}^n$, i.e. $\pi_j x = (x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_N)$, and define its adjoint action on the function $\varphi(x) : \mathbb{Z}^N \to \mathbb{C}$ by: $\pi_j \varphi(x) = \varphi(\pi_j x), j = 1, \dots, N$. Let $\mathbb{C}[[\xi]]$ be the ring of formal power series in the variable $\xi = (\xi_1, \dots, \xi_N)$ and define the operator $\pi_j : \mathbb{C}[[\xi]] \to \mathbb{C}[[\xi]]$ for $j = 1, \dots, N$ on the generating series $\Phi(\xi) = \sum\limits_{x \in \mathbb{Z}^N} \varphi(x) \xi^x$ as follows

$$\pi_j \Phi(\xi) = \sum_{x \in \mathbb{Z}^N} \varphi(\pi_j x) \xi^{\pi_j x} = \Phi(\xi_1, \dots, \xi_{j-1}, 0, \xi_{j+1}, \dots, \xi_N).$$

Furthermore, for $J \in V$ let $\pi_J = \pi_{j_1} \circ \dots \circ \pi_{j_k}$ be a composition of operators $\pi_{j_1}, \dots, \pi_{j_k}$ and $\pi_\emptyset = 1$ is the the identity operator.

For complex-valued functions $\varphi(x)$ of integer arguments $x = (x_1, \dots, x_N)$ we define a shift operator δ_j for $j = 1, \dots, N$ as follows

$$\delta_j \varphi(x) = \varphi(x_1, \dots, x_{j-1}, x_j + 1, x_{j+1}, \dots, x_N)$$

and for $\mu = (\mu_1, \dots, \mu_N) \in \mathbb{Z}^N$ we define $\delta^\mu = \delta_1^{\mu_1} \cdots \delta_N^{\mu_N}$.

For $c = (c_1, \dots, c_N) \in \mathbb{C}^N$ we denote the operator $Q(\delta) = \delta_1 \cdot \ldots \cdot \delta_N - c_1 \delta_2 \cdot \ldots \cdot \delta_N - \dots - c_N \delta_1 \cdot \ldots \cdot \delta_{N-1}$ and $z^A = (z^{\alpha^1}, \dots, z^{\alpha^N})$.

Now we formulate a multidimensional analogue of identity (2.5).

Theorem 2.2. *Let $P_A(\lambda;\varphi)$ be the vector partition function, associated with a function $\varphi : \mathbb{Z}^N_{\geqslant} \to \mathbb{C}$, and $\Phi(\xi)$ be the generating series for $\varphi(x)$. Then*

$$\sum_{J\in V}(-1)^{\#J}\pi_J\left[(1-\langle c,\xi\rangle)\Phi(\xi)\right]\Big|_{\xi=z^A} = \sum_{\lambda\in K\cap\mathbb{Z}^N} P_A(\lambda;Q(\delta)\varphi)z^{\lambda}. \qquad (2.8)$$

We note that for $N=1$ and $c=1$, formula (2.8) implies identity (2.5).

The problem of finding *the number of generalized lattice paths* is formulated as follows: find the number of paths on an integer lattice from the origin to the point $\lambda\in\mathbb{Z}^n$ using only steps in Δ.

A similar problem for $n=2$ connected with the study of multidimensional difference equations and its application for generalized Dyck paths (see [27]) was considered in [8]. We note that the Cauchy problem for multidimensional difference equations was considered also in [29], [34] and some properties of generating function of its solution were investigated in [30], [46], [47].

A simple case of the problem above arises by choosing the set of steps which form an orthonormal basis in $\mathbb{R}^N$: $\alpha^j = e^j, j=1,\dots,N$, and the number $\varphi(x)$ of paths from the origin to the point $x\in\mathbb{Z}^N$ satisfies the basic recurrence relation of combinatorial analysis

$$(1-\langle I,\delta^{-I}\rangle)\varphi(x)\equiv\varphi(x)-\varphi(x-e^1)-\cdots-\varphi(x-e^N)=0 \qquad (2.9)$$

where $x\in I+\mathbb{Z}^N_{\geqslant}$. For any solution of (2.9) we have:

Proposition 2.3. *If the function $\varphi(x)$ satisfies equation* (2.9)*, then the associated vector partition function $P_A(\lambda;\varphi)$ satisfies the difference equation*

$$(1-\langle I,\delta_{\lambda}^{-A}\rangle)P_A(\lambda;\varphi)\equiv P_A(\lambda;\varphi)-\sum_{j=1}^{N}P_A(\lambda-\alpha^j;\varphi)=0.$$

The following proposition relates the number of lattice paths $\varphi(x)$ and the number of generalized lattice paths $P_A(\lambda;\varphi)$:

Proposition 2.4. *If $\varphi(x)$ is the number of lattice paths, then the associated the vector partition function $P_A(\lambda;\varphi)$ coincides with the number of generalized lattice paths with steps in Δ; in this case its generating function $F(z) = \sum\limits_{\lambda\in K\cap\mathbb{Z}^n} P_A(\lambda;\varphi)z^\lambda$ has the form*

$$F(z) = \frac{1}{1 - z^{\alpha^1} - z^{\alpha^2} - \cdots - z^{\alpha^N}}.$$

2.3 Chaundy-Bullard identity

In 1960 T. Chaundy and J. Bullard in [10] considered the identity which is valid for $c_1, c_2 \in \mathbb{C}$ such that $c_1 + c_2 = 1$ and nonnegative integers μ_1 and μ_2

$$c_2^{\mu_2+1}\sum_{\nu_1=0}^{\mu_1}\binom{\mu_1+\mu_2-\nu_1}{\mu_1-\nu_1}c_1^{\mu_1-\nu_1} + c_1^{\mu_1+1}\sum_{\nu_2=0}^{\mu_2}\binom{\mu_1+\mu_2-\nu_2}{\mu_2-\nu_2}c_2^{\mu_2-\nu_2} \equiv 1.$$

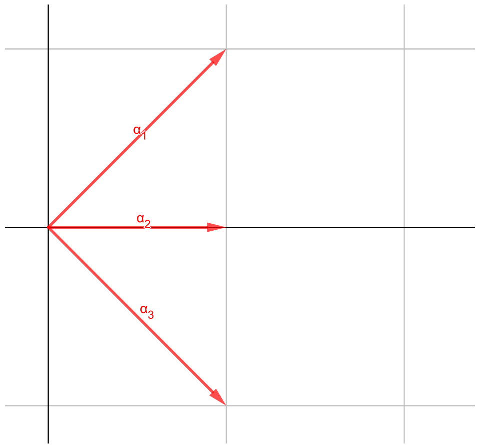
This identity was subsequently found in approximation theory, nonrecursive digital filters [20], in the theory of wavelets [11], in the theory of Gauss hypergeometric functions. A detailed review, including various proofs of a one-dimensional case and a multidimensional analogues of this identity was given in [23], [24] and [14]. In [25] similar identities were derived by using methods of generating functions and properties of Hadamards composition of multiple power series.

We give an analogue of the Chaundy-Bullard identity for vector partition functions.

For $j = 1,\ldots,N$ we denote $\Delta_j = \Delta\backslash\{\alpha^j\}$ and $A_j = [\alpha^1,\ldots,\alpha^{j-1},\alpha^{j+1},\ldots,\alpha^N]$, then $K_j = \{\nu\in\mathbb{Z}^n : \nu = y_1\alpha^1 + \ldots[j]\ldots + y_N\alpha^N, y\in\mathbb{Z}^N_{\geqslant}\}$; $c^x = c_1^{x_1}\cdots c_N^{x_N}$. Note that each cone $K_j\subset K$ is also pointed.

Theorem 2.5. *If $c_1 + c_2 + \cdots + c_N = 1$ and $\varphi_j(x) = \frac{|x|!}{x!}\,c^{x+e^j}$, then for any $\mu = (\mu_1,\ldots,\mu_N)\in\mathbb{Z}^N$ the identity*

$$\sum_{j=1}^{N}\sum_{\nu\in K_j} P_{A_j}(\nu)P_A(\mu-\nu;\varphi_j) = P_A(\mu) \tag{2.10}$$



takes place.

The sum on the left side of identity (2.10) is finite since all the cones $K_j, j = 1, \ldots, N$ are pointed.

Note that for $\alpha^j = e^j, j = 1, \ldots, N$ we obtain from (2.10) a multidimensional Chaundy-Bullard identity:

$$\sum_{j=1}^{N} \sum_{\substack{0 \leqslant \nu \leqslant \mu \\ \nu_j = 0}} \frac{(|\mu| - |\nu|)!}{(\mu - \nu)!} c^{\mu - \nu + e^j} \equiv 1,$$

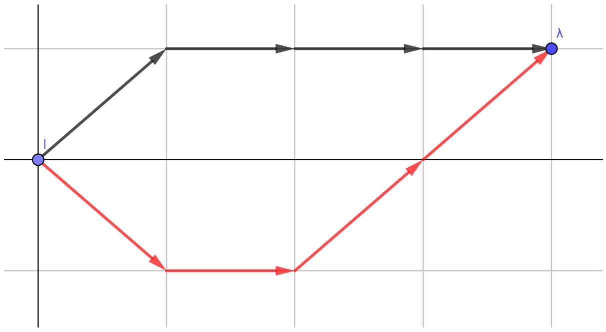
where the double inequality $0 \leqslant \nu \leqslant \mu$ means that $0 \leqslant \nu_j \leqslant \mu_j$ for all $j = 1, \ldots, N$.

Example. Let's consider a set of 3 steps: $\Delta = \{\alpha^1 = (1,1), \alpha^2 = (1,0), \alpha^3 = (1,-1)\}$.

Then for $\lambda = (4,1)$, $\varphi(x) = 1$ we have $P_A(\lambda; \varphi) = P_A(\lambda)$ is a number of non-negative integer solutions to the system

$$\begin{cases} x_1 + x_2 + x_3 & = 4 \\ x_1 - x_3 & = 1 \end{cases}.$$

It has 2 solutions $(1,3,0)$ and $(2,1,1)$, which also means that we can get from the origin to the point $\lambda = (4,1)$ using 2 different sequences of the steps: $\alpha_1\alpha_2\alpha_2\alpha_2$ and $\alpha_1\alpha_1\alpha_2\alpha_3$.



This allows us to compute the number of paths from the origin to the point λ:

$$f(\lambda) = C_4^1 C_3^3 + C_4^2 C_2^1 C_1^1.$$

For $c_1 + c_2 + c_3 = 1$ we get the Chaundy-Bullard identity

$$\begin{aligned} c_1(12c_1^2c_2c_3 + 4c_1c_2^3 + 3c_1^2c_2 + 3c_1^2c_3 + 3c_1c_2^2 + c_1^2 + 2c_1c_2 + c_1)+ \\ + c_2(12c_1^2c_2c_3 + 4c_1c_2^3 + 3c_1^2c_2 + 6c_1c_2c_3 + c_2^3 + 2c_1c_2 + 2c_2c_3 + c_2)+ \\ c_3(12c_1^2c_2c_3 + 4c_1c_2^3 + 3c_1^2c_3 + 3c_1c_2^2 + 6c_1c_2c_3 + c_2^3+ \\ 2c_1c_2 + 2c_1c_3 + c_2^2 + 2c_2c_3 + c_1 + c_2 + c_3 + 1) = 2. \end{aligned}$$

2.4 Proofs

In this section we prove the main result (see Theorem 2.2) and its corollaries. First, using Lemmas 2.6 and 2.7, we prove the identity (2.8) for a set of standard basis vectors, and then, using a monomial substitution, we prove Theorem 2.2. The proofs of Propositions 2.3 and 2.4 follow directly from the main result. Then, using Lemmas 2.8 and 2.9, we prove Theorem 2.5.

We prove the identity (2.8) for the case when the set of vectors $\Delta = \{\alpha^1, \dots, \alpha^N\}$ consists of unit vectors $\alpha^j = e^j$, where the vector $e^j = (0, \dots, 0, 1, 0, \dots, 0)$ contains a unit on the j-th place for $j = 1, \dots, N$. Then the vector partition function $P_A(\lambda; \varphi) = \varphi(x)$, and the generating series $\Phi(\xi) = \sum\limits_{x \in \mathbb{Z}_{\geqslant}^N} \varphi(x)\xi^x$ in identity (2.8) takes the form

$$\sum_{J\in V}(-1)^{\#J}\pi_J\left[(1-\langle c,\xi\rangle)\Phi(\xi)\right]=\sum_{x\in I+\mathbb{Z}^N_{\geqslant}}(1-\langle c,\delta^{-I}\rangle)\varphi(x)\xi^x, \tag{2.11}$$

where $\langle c,\xi\rangle=c_1\xi_1+\cdots+c_N\xi_N$, and $I=(1,\ldots,1)$.

To prove (2.11) we use following properties of the operator $\mathbf{\Pi}=\sum\limits_{J\in V}(-1)^{\#J}\pi_J$:

Lemma 2.6. *For an arbitrary series* $\Phi(\xi)=\sum\limits_{x\in\mathbb{Z}^N_{\geqslant}}\varphi(x)\xi^x$ *the operator* $\mathbf{\Pi}$ *acts on* $\Phi(\xi)$ *as follows*

$$\mathbf{\Pi}:\sum_{x\in\mathbb{Z}^N_{\geqslant}}\varphi(x)\xi^x\mapsto\sum_{x\in I+\mathbb{Z}^N_{\geqslant}}\varphi(x)\xi^x.$$

Proof. We represent the operator $\mathbf{\Pi}$ as a composition $\mathbf{\Pi}=(1-\pi_1)(1-\pi_2)\cdots(1-\pi_N)$, where $1=\pi_\emptyset$ is an identity operator, and use the commutativity of its factors to apply it to the series $\Phi(\xi)$:

$$\begin{aligned}
&(1-\pi_1)(1-\pi_2)\cdots(1-\pi_N)\Phi(\xi)=\\
&\qquad=(1-\pi_1)(1-\pi_2)\cdots(1-\pi_{N-1})\left[\Phi(\xi)-\Phi(\pi_N\xi)\right]=\\
&\qquad=(1-\pi_1)(1-\pi_2)\cdots(1-\pi_{N-1})\sum_{x\in e^N+\mathbb{Z}^N_{\geqslant}}\varphi(x)\xi^x=\\
&\qquad=(1-\pi_1)(1-\pi_2)\cdots(1-\pi_{N-2})\sum_{x\in e^N+e^{N-1}+\mathbb{Z}^N_{\geqslant}}\varphi(x)\xi^x=\\
&\qquad=\ldots=\sum_{x\in I+\mathbb{Z}^N_{\geqslant}}\varphi(x)\xi^x,
\end{aligned}$$

where $I=e^1+e^2+\ldots+e^N=(1,1,\ldots,1)$. $\square$

Lemma 2.7. *If* $\mathbf{\Pi}_j=(1-\pi_1)\cdots(1-\pi_{j-1})(1-\pi_{j+1})\cdots(1-\pi_N)$, *then for any* $j=1,\ldots,N$ *the following equality holds*

$$\mathbf{\Pi}\xi_j\Phi(\xi)=\mathbf{\Pi}_j\xi_j\Phi(\xi)=\sum_{x\in I+\mathbb{Z}^N_{\geqslant}}\varphi(x-e^j)\xi^x.$$

Proof. Similarly as in the proof of Lemma 2.6 we represent the operator $\mathbf{\Pi}$ as a composition and apply it to $\xi_j\Phi(\xi) \in \mathbb{C}[[\xi]]$:

$$
\begin{aligned}
&(1-\pi_1)(1-\pi_2)\cdots(1-\pi_N)\left[\xi_j\Phi(\xi)\right] = \\
&\quad = (1-\pi_1)\cdots(1-\pi_{j-1})(1-\pi_{j+1})\cdots(1-\pi_N)\left[(1-\pi_j)\xi_j\Phi(\xi)\right] = \\
&\quad = (1-\pi_1)\cdots(1-\pi_{j-1})(1-\pi_{j+1})\cdots(1-\pi_N)\left[\xi_j\Phi(\xi) - \pi_j\xi_j\Phi(\xi)\right] = \\
&\quad = (1-\pi_1)\cdots(1-\pi_{j-1})(1-\pi_{j+1})\cdots(1-\pi_N)\left[\xi_j\Phi(\xi)\right] = \\
&\quad = \mathbf{\Pi}_j \sum_{x\in e^j+\mathbb{Z}^N_{\geqslant}} \varphi(x-e^j)\xi^x = \sum_{x\in I+\mathbb{Z}^N_{\geqslant}} \varphi(x-e^j)\xi^x.
\end{aligned}
$$

□

By using Lemma 2.6 and 2.7.

Proof of (2.10). We apply the operator $\mathbf{\Pi}$ to the product $(1-\langle c,\xi\rangle)\Phi(\xi)$ and use Lemmas 2.6 and 2.7 to obtain:

$$
\begin{aligned}
&\mathbf{\Pi}\left[(1-\langle c,\xi\rangle)\Phi(\xi)\right] = \mathbf{\Pi}\Phi(\xi) - \mathbf{\Pi}\left[\langle c,\xi\rangle\Phi(\xi)\right] = \mathbf{\Pi}\Phi(\xi) - \langle c,\mathbf{\Pi}\xi\rangle\Phi(\xi) = \\
&\quad = \mathbf{\Pi}\Phi(\xi) - c_1\mathbf{\Pi}_1\xi_1\Phi(\xi) - \cdots - c_N\mathbf{\Pi}_N\xi_N\Phi(\xi) = \\
&\quad = \sum_{x\in I+\mathbb{Z}^N_{\geqslant}} \varphi(x)\xi^x - c_1\sum_{x\in I+\mathbb{Z}^N_{\geqslant}} \varphi(x-e^1)\xi^x - \ldots - c_N\sum_{x\in I+\mathbb{Z}^N_{\geqslant}} \varphi(x-e^N)\xi^x = \\
&\quad = \sum_{x\in I+\mathbb{Z}^N_{\geqslant}} \left[\varphi(x) - c_1\varphi(x-e^1) - \ldots - c_N\varphi(x-e^N)\right]\xi^x = \\
&\quad = \sum_{x\in I+\mathbb{Z}^N_{\geqslant}} \left[(1-\langle c,\delta^{-I}\rangle)\varphi(x)\right]\xi^x.
\end{aligned}
$$

□

Proof of Theorem 2.2. We substitute the monomial replacement of the variables $\xi = z^A$ in the formula (2.11) to obtain

$$
\xi^x = (z_1^{\alpha_1^1}\cdots z_n^{\alpha_n^1})^{x_1}\ldots(z_1^{\alpha_1^N}\cdots z_n^{\alpha_n^N})^{x_N} = z_1^{x_1\alpha_1^1+\ldots+x_N\alpha_1^N}\cdots z_n^{x_1\alpha_n^1+\ldots+x_N\alpha_n^N} = z^\lambda,
$$

where $\lambda = Ax$. We further observe that if $x \in I + \mathbb{Z}^N_{\geqslant}$, then $\lambda \in K \cap \mathbb{Z}^n$. Therefore,

$$\sum_{J \in V} (-1)^{\# J} \pi_J \left[(1 - \langle c, \xi \rangle)\Phi(\xi)\right] \Big|_{\xi = z^A} =$$

$$= \sum_{x \in I + \mathbb{Z}^N_{\geqslant}} \left[\varphi(x) - c_1 \varphi(x - e^1) - \ldots - c_N \varphi(x - e^N)\right] \xi^x \Big|_{\xi = z^A} =$$

$$= \sum_{\lambda \in K \cap \mathbb{Z}^n} \sum_{\substack{Ax = \lambda \\ x \in \mathbb{Z}^N_{\geqslant}}} \left[\varphi(x + I) - c_1 \varphi(x + I - e^1) - \ldots - c_N \varphi(x + I - e^N)\right] z^\lambda =$$

$$= \sum_{\lambda \in K \cap \mathbb{Z}^n} P_A\left(\lambda; Q(\delta)\varphi\right) z^\lambda,$$

where $Q(\delta) = \delta_1 \cdot \ldots \cdot \delta_N - c_1 \delta_2 \cdot \ldots \cdot \delta_N - \cdots - c_N \delta_1 \cdot \ldots \cdot \delta_{N-1}$. □

Proof of Proposition 2.3. Summing the left side of the basic recurrence relation (2.9) over all integer nonnegative $x : Ax = \lambda$, we obtain

$$\sum_{x:Ax=\lambda} \varphi(x) - \sum_{x:Ax=\lambda} \varphi(x - e^1) - \cdots - \sum_{x:Ax=\lambda} \varphi(x - e^N) = 0. \qquad (2.12)$$

Note that for any $j = 1, \ldots, N$ we have

$$\sum_{\substack{x:Ax=\lambda \\ x \geqslant 0}} \varphi(x - e^j) = \sum_{\substack{x:A(x - e^j + e^j) = \lambda \\ x \geqslant 0}} \varphi(x - e^j) = \sum_{\substack{x:A(x - e^j) = \lambda - \alpha^j \\ x \geqslant 0}} \varphi(x - e^j) =$$

$$= \sum_{\substack{x:Ax=\lambda-\alpha^j \\ x \geqslant -e^j}} \varphi(x) = \sum_{\substack{x:Ax=\lambda-\alpha^j \\ x \geqslant 0}} \varphi(x) - \sum_{\substack{x:Ax=\lambda-\alpha^j \\ x_j = -e^j}} \varphi(x) =$$

$$= P_A(\lambda - \alpha^j; \varphi),$$

since $\varphi(x) = 0$ for all $x \notin \mathbb{Z}^N_{\geqslant}$. Then from (2.12) we obtain

$$P_A(\lambda; \varphi) - P_A(\lambda - \alpha^1; \varphi) - \ldots - P_A(\lambda - \alpha^N; \varphi) = 0.$$

□

Proof of Proposition 2.4. We prove that the generating function $\Phi(\xi)$ of the number of lattice paths is equal to $\Phi(\xi) = (1 - \xi_1 - \xi_2 - \ldots - \xi_N)^{-1}$. Indeed, the number

$\varphi(x)$ of lattice paths satisfies the basic recurrence relation (2.9). Therefore, the right side of (2.11) vanishes. Now we use induction on the number N of variables $\xi = (\xi_1, \xi_2, \dots, \xi_N)$. For $N = 1$ formula (2.11) takes the form $(1-\xi)\Phi(\xi) - 1 = 0$, where $\Phi(\xi) = (1-\xi)^{-1}$ and for any number $m < N$ of variables $(\xi_{i_1}, \xi_{i_2}, \dots, \xi_{i_m})$ the generating function $\Phi(\xi_{i_1}, \xi_{i_2}, \dots, \xi_{i_m}) = (1 - \xi_{i_1} - \xi_{i_2} - \dots - \xi_{i_m})^{-1}$.

We note that for any $J = \{j_1, j_2, \dots, j_k\}, k \leqslant N, J \neq \emptyset$ the number of lattice paths in $\mathbb{Z}^{N-k} = \mathbb{Z}^N \cap \{x_{j_1} = \dots = x_{j_k} = 0\}$ can be written as a function $\pi_J \varphi(x)$, and the induction hypothesis implies that its generating function $\Phi(\xi)$ satisfies the relation $\pi_J \Phi(\xi) = \pi_J (1 - \xi_1 - \dots - \xi_N)^{-1}$ or $\pi_J \left[(1 - \xi_1 - \dots - \xi_N)\Phi(\xi)\right] = 1$.

Next, we select in (2.11) the term corresponding to $J = \emptyset$:

$$(1 - \xi_1 - \ldots - \xi_N)\Phi(\xi) + \sum_{\substack{J \neq \emptyset \\ J \in V}} (-1)^{\#J} 1 = 0.$$

The equality $\sum\limits_{\substack{J \neq \emptyset \\ J \in V}} (-1)^{\#J} 1 = -C^N_{N-1} + C^N_{N-2} + \dots + (-1)^N C^N_0 = -1$ implies that the induction statement is also true for $m = N$. After making the substitution $\xi = z^A$ we obtain Proposition 2.4. □

The following two lemmas are required to prove the Theorem 2.5.

Lemma 2.8. *If the function $\Phi(\xi)$ does not depend on the variable ξ_j, then $\mathbf{\Pi}\Phi(\xi) = 0$.*

Proof. We consider the operator $\mathbf{\Pi}_j = (1 - \pi_1) \circ \cdots [j] \cdots \circ (1 - \pi_N), j = 1, \dots, N$. Since the function $\Phi(\xi)$ does not depend on the variable ξ_j, we have $(1 - \pi_j)\Phi(\xi) = \Phi(\xi) - \pi_j \Phi(\xi) = 0$, therefore $\mathbf{\Pi}\Phi(\xi) = \mathbf{\Pi}_j (1 - \pi_j)\Phi(\xi) = 0$. □

Lemma 2.9. *For any complex values $c_1, \dots, c_N$ such that $c_1 + \dots + c_N = 1$, the rational fraction $(I - \xi)^{-1}$ can be represented as follows:*

$$\frac{1}{I - \xi} = \sum_{j=1}^{N} \frac{c_j}{\pi_j (I - \xi)} \cdot \frac{1}{1 - \langle c, \xi \rangle}. \tag{2.13}$$

Proof. Since $c_1 + \cdots + c_N = 1$ and $\varphi(x) \equiv 1$, the right side of (2.11) vanishes, and its left side can be written as follows:

$$(1 - \langle c, \xi \rangle)\frac{1}{I - \xi} + \sum_{J \neq \emptyset}(-1)^{\# J}\pi_J(1 - \langle c, \xi \rangle)\Phi(\xi) = 0.$$

Since $\sum\limits_{J \neq \emptyset}(-1)^{\# J}\pi_J = \mathbf{\Pi} - 1$ and $(1 - \langle c, \xi \rangle) = \sum\limits_{j=1}^{N} c_j(1 - \xi_j)$, we obtain

$$\frac{1 - \langle c, \xi \rangle}{I - \xi} + \sum_{j=1}^{N} \mathbf{\Pi}\frac{c_j}{\pi_j(I - \xi)} = \sum_{j=1}^{N} \frac{c_j}{\pi_j(I - \xi)}.$$

Furthermore, Lemma 2.8 implies

$$\mathbf{\Pi}\left(\pi_j \frac{c_j}{I - \xi}\right) = 0, j = 1, \ldots, N,$$

which completes the proof. $\square$

Proof of Theorem 2.5. We expand both sides of (2.13) in power series in ξ^μ to obtain

$$\sum_{\mu \in \mathbb{Z}^N_{\geqslant}} \xi^\mu = \sum_{j=1}^{N} \frac{c_j}{\pi_j(I - \xi)} \sum_{\mu \in \mathbb{Z}^N_{\geqslant}} \frac{c^\mu |\mu|!}{\mu!}\, \xi^\mu. \tag{2.14}$$

We recall that $K = \{\mu \in \mathbb{Z}^n : \mu = x_1\alpha^1 + \ldots + x_N\alpha^N, x \in \mathbb{Z}^N_{\geqslant}\}$, $K_j = \{\nu \in \mathbb{Z}^n : \nu = y_1\alpha^1 + \ldots [j] \ldots + y_N\alpha^N, y \in \mathbb{Z}^N_{\geqslant}\}$, and $K_j \subset K$ for $j = 1, \ldots, N$. In (2.14) we substitute $\xi = z^A$ and transform the left side as follows

$$\frac{1}{I - z^A} = \sum_{x \in \mathbb{Z}^n_{\geqslant}} z^{Ax} = \sum_{\mu \in K}\left(\sum_{\substack{x : Ax = \mu \\ x \in \mathbb{Z}^n_{\geqslant}}} 1\right) z^\lambda = \sum_{\mu \in K} P_A(\mu) z^\mu.$$

We denote $\varphi_j(x) = \frac{|x|!}{x!}\, c^{x+e^j}$, then the right side of (2.14) takes the form

$$\sum_{j=1}^{N} c_j \pi_j (I-\xi)^{-1}(1-\langle c,\xi\rangle)^{-1} = \sum_{j=1}^{N} \left(\sum_{\substack{y\geqslant 0\\ y_j=0}} \xi^y \cdot \sum_{x\in\mathbb{Z}^N_{\geqslant}} \varphi_j(x) \right) =$$

$$= \sum_{j=1}^{N} \left(\sum_{\nu\in K_j} \left(\sum_{\substack{y:Ay=\nu\\ y_j=0}} 1 \right) z^\nu \cdot \sum_{\lambda\in K} \left(\sum_{x:Ax=\lambda} \varphi_j(x) \right) z^\lambda \right) =$$

$$= \sum_{j=1}^{N} \left(\sum_{\nu\in K_j} P_{\Delta_j}(\nu) z^\nu \cdot \sum_{\lambda\in K} P_\Delta(\lambda;\varphi_j(x)) z^\lambda \right) =$$

$$= \sum_{\mu\in K} \left(\sum_{j=1}^{N} \sum_{\substack{\nu+\lambda=\mu\\ \nu\in K_j\\ \lambda\in K}} P_{\Delta_j}(\nu) P_\Delta(\lambda;\varphi_j(x)) \right) z^\mu.$$

Equating the coefficients of z^μ yields

$$P_A(\mu) = \sum_{j=1}^{N} \sum_{\substack{\nu+\lambda=\mu\\ \nu\in K_j\\ \lambda\in K}} P_{A_j}(\nu) P_A(\lambda;\varphi_j(x)) = \sum_{j=1}^{N} \sum_{\nu\in K_j} P_{A_j}(\nu) P_A(\mu-\nu;\varphi_j(x)),$$

which completes the proof. □

Chapter 3. Difference equations and generating functions for some lattice path problems

In chapter 3 we consider a difference equation in a two-dimensional pointed lattice cone K spanned by a set of vectors including n linearly independent vectors. The appropriate substitution of variables yields the difference equations in the positive octant $\mathbb{Z}^2_{\geqslant}$, which allows us to use methods for studying the Cauchy problem developed in [8], [35], [57]. Moreover, we managed to find a linear difference equation with non-constant coefficients for the restricted lattice paths problem and solved it using properties of diagonal power series.

3.1 Definitions and notations

We let $\mathbb{Z}, \mathbb{Z}_{\geqslant}, \mathbb{Q}, \mathbb{R}, \mathbb{R}_{\geqslant}, \mathbb{C}$ denote the integer, nonnegative integer, rational, real, nonegative real, and complex numbers. For each positive integer N, $\mathbb{R}^N$ is N-dimensional real vector space and its discrete subgroup $\mathbb{Z}^N$ is the *standard N-dimensional lattice*. We let 0 denote the origin in $\mathbb{Z}^N$ and $\delta_0 : \mathbb{Z}^N \to \mathbb{C}$ denote the function such that $\delta_0(0) = 1$ and $\delta_0(x) = 0$ for $x \neq 0$.

For $z = (z_1, \ldots, z_N)$ where $z_i, i = 1, \ldots, N$ are indeterminates, we let $\mathbb{C}[z]$, $\mathbb{C}(z)$, $\mathbb{C}[[z]]$ denote the ring of polynomials, the field of rational functions, and the ring of formal power series in $z_1, ..., z_N$. If $f : \mathbb{Z}^N_{\geqslant} \to \mathbb{C}$ we define its generating function $F(z) \in \mathbb{C}[[z]]$

$$F(z) = \sum_{x \in \mathbb{Z}^N_{\geqslant}} f(x)\, z^x \tag{3.1}$$

where $z^x = z_1^{x_1} \cdot \ldots \cdot z_N^{x_N}$. The correspondence $f \to F$ gives a bijection between functions on $\mathbb{Z}^N_{\geqslant}$ and formal power series. We will use lower case letters f, g, h, φ to denote functions on $\mathbb{Z}^N_{\geqslant}$ and corresponding upper case letters F, G, H, Φ to denote their generating functions. If f is a function on $\mathbb{Z}^N_{\geqslant}$ we will identify it as a function on $\mathbb{Z}^N$ by setting it equal to zero on the complement $\mathbb{Z}^N \backslash \mathbb{Z}^N_{\geqslant}$.

A linear finite difference equation is an equation having the form

$$\sum_{y \in S} c_y(x) f(x-y) = g(x), \;\; x \in \mathbb{Z}^N, \tag{3.2}$$

where $S \subset \mathbb{Z}^N$ is finite, $c_y : \mathbb{Z}^N \to \mathbb{C}$ are a set of *coefficient functions*, and $g : \mathbb{Z}^N \to \mathbb{C}$. A solution of equation (3.2) is a function $f : \mathbb{Z}^N \to \mathbb{C}$ that satisfies the equation. The equation is *homogeneous* if $g = 0$. The set of solutions of a homogeneous equation is a vector space over the field $\mathbb{C}$.

The equation has constant coefficients if each coefficient function is constant. In this paper we make extensive use of the relationship between finite difference equations and generating functions. We now illustrate this relationship using the Fibonacci sequence for which $N = 1$ and $f : \mathbb{Z}^N_{\geqslant} \to \mathbb{C}$ defined by

$$f(0) = f(1) = 1, \; f(x) = f(x-1) + f(x-2), \; x \geqslant 2.$$

Now regard f a function on $\mathbb{Z}^N$ and define $g = \delta_0$, $S = \{0, 1, 2\}$, and constant coefficient functions $c_0 = 1, c_1 = c_2 = -1$. Then f satisfies equation (3.2). Multiply both sides of this equation by the monomial z^x and sum over $x \in \mathbb{Z}^N_{\geqslant}$ to obtain

$$\sum_{x \in \mathbb{Z}^N_{\geqslant}} \sum_{y \in S} c_y(x) f(x-y) z^x = \sum_{x \in \mathbb{Z}^N_{\geqslant}} g(x) z^x = 1. \tag{3.3}$$

The left side of equation (3.3) is $(1 - z - z^2)F(z)$, so partial fractions gives

$$F(z) = \frac{1}{1 - z - z^2} = \frac{-\sqrt{5}}{5(z - \mu_+)} + \frac{\sqrt{5}}{5(z - \mu_-)} \tag{3.4}$$

where $\mu_{\pm} = \frac{-1 \pm \sqrt{5}}{2}$. and the geometric series gives

$$f(x) = \alpha_+ \, \lambda_+^x + \alpha_- \, \lambda_-^x \tag{3.5}$$

where $\lambda_{\pm} = \frac{1 \pm \sqrt{5}}{2}$ and $\alpha_{\pm} = \frac{5 \pm \sqrt{5}}{10}$.

Let $\ell \geqslant 0$. A *lattice path with length* ℓ is a finite sequence $p(0), p(1), \ldots, p(L)$ of *points* in $\mathbb{Z}^N$, and its *steps* are the set of lattice vectors $\{0\} \cup \{p(k) - p(k-1) : k = 1, \ldots, \ell\}$. Specific *classes* of lattice paths arise by placing conditions on the paths including: the steps are in a specified $S \subset \mathbb{Z}^N$, the points are in a specified $P \subset \mathbb{Z}^N$, fixing the length L, and requiring that the points are distinct (this describes non intersecting paths).

A lattice path *counting problems* compute the function $f : \mathbb{Z}^N \to \mathbb{Z}_{\geqslant}$ that counts the number $f(x)$ of paths, in a specified class, for which $p(0) = 0$ and $p(L) = x$ (the condition that $p(0) = 0$ does not result in a loss of generality).

If the class (of lattice paths) is constrained only by specifying the set S of possible paths then f satisfies the *finite difference equation*

$$f(x) = \sum_{y \in S} f(x - y). \tag{3.6}$$

This equation admits a solution iff S satisfies condition A, and then f is the unique solution of Equation (3.7) that satisfies $f(0) = 1$.

A general linear homogeneous finite difference equation has the form

$$f(x) = \sum_{y \in S} c(x, y)\, f(x - y) \tag{3.7}$$

where S is finite, $f : \mathbb{Z}^N \to \mathbb{C}$ and the coefficients $c(x, y) \in \mathbb{C}$. If $N = 1$, and each $c(x, y), y \in S$ is a rational function of x, then f is a *holonomic* function.

3.2 Restricted lattice class problems

We will prove the identity for generating functions based on which we develop a novel method to compute the number of restricted lattice paths. This method exploit a difference equation with non-constant coefficients.

Restricted lattice class problems compute f for a class of paths whose points belong to a specified subset $P \subset \mathbb{Z}^N$. Clearly $0 \in P$ else there are no paths in P

that start at 0. If the possible set of steps $S \subset \mathbb{Z}^N$ then the counting function f has support in P, $f(0) = 1$, and f satisfies the linear homogeneous difference equation

$$f(x) = \sum_{y \in S} \chi_P(x)\, f(x - y), \tag{3.8}$$

where χ_P is the *characteristic function* of P. We recall that $\chi_P(x) = 1$ if $x \in P$ and $\chi_P(x) = 0$ if $x \notin P$. The solution f will also satisfy equation (3.7) iff the set of lattice points in the cone spanned by S is a subset of P

In this paper we find f for selected classes of lattice paths. For all of these classes $S \subset \mathbb{Z}^N_{\geqslant}$ so f is supported on $\mathbb{R}^N_{\geqslant}$ and therefore f is uniquely represented by its *generating function* $F(z) \in \mathbb{C}[[z]]$ (the ring of formal power series in $z = (z_1, \ldots, z_N)$ where $z_i, i = 1, \ldots, N$ are indeterminates) which is defined by

$$F(z) = \sum_{x \in \mathbb{Z}^N_{\geqslant}} f(x)\, z^x. \tag{3.9}$$

Our method employs a difference equation with non-constant coefficients for $f(x)$ to compute its generating function $F(z)$. We illustrate this method by counting Dyck paths, Schröder paths, Motzkin paths, and more general paths. For these cases the terms in $F(z)$ corresponding to the terms with non-constant coefficients in the difference equation for $f(x)$ are the generating function of a diagonal subsequence of $f(x)$.

Let $\mathbb{Z}$ be the set of integers numbers, and $\mathbb{Z}_{\geq}$ is the set of non-negative integers. We consider the difference equation of the form

$$\varphi(x) = c_1\varphi(x - e^1) + \cdots + c_N\varphi(x - e^N),\ x \in \mathbb{Z}^N \tag{3.10}$$

where $\mathbb{Z}^N = \mathbb{Z} \times \cdots \times \mathbb{Z}$, which we will call the basic recurrence relation, since for $n = 2$ and $c_1 = c_2 = 1$ its solutions are binomial coefficients. We consider the mapping $A : \mathbb{Z}^N \to \mathbb{Z}^N_{\geq}$ with matrix

$$A = \begin{pmatrix} \alpha_1^1 & \dots & \alpha_1^N \\ \dots & \dots & \dots \\ \alpha_n^1 & \dots & \alpha_n^N \end{pmatrix}_{n\times N},$$

whose elements are $\alpha_i^j \in \mathbb{Z}$. We denote by $\alpha^j = Ae^j$ the columns of this matrix and consider the difference equation

$$f(\lambda) = \sum_{j=1}^{N} c_j f(\lambda - \alpha^j),\ \lambda \in \mathbb{Z}^n. \tag{3.11}$$

We assume that these are the first n vectors, then any $\lambda \in K$ can be represented in a unique way in the form $\lambda = \nu_1\alpha^1 + \dots + \nu_n\alpha^n,\ \nu_j \in \mathbb{Z}_{\geqslant}, j = 1, \dots, n$, and the vectors $\alpha^{n+1}, \dots, \alpha^N$ in the form $\alpha^{n+1} = \beta_1^{n+1}\alpha^1 + \dots + \beta_n^{n+1}\alpha^n, \dots, \alpha^N = \beta_1^N\alpha^1 + \dots + \beta_n^N\alpha^N, \beta_j^i \in \mathbb{Z}_{\geqslant}$. If $g(\nu) = f(\nu_1\alpha^1 + \dots + \nu_n\alpha^n)$, then the difference equation takes the form

$$g(\nu) = c_1 g(\nu-\beta^1)+\dots+c_n g(\nu-\beta^n)+c_{n+1} g(\nu-\beta^{n+1})+\dots+c_N g(\nu-\beta^N), \tag{3.12}$$

where $\beta^j = (\beta_1^j, \dots, \beta_n^j), j = n+1, \dots, N$.

The method described above from difference equations of the form (3.11) to difference equations of the form (3.12) can be used in the problem of counting lattice paths in the two-dimensional integer lattice. In particular, it is useful in classical problems for Dyck, Motzkin and Schröder paths.

In the case of arbitrary n, this allows the use of methods developed in [8] to find generating functions for solving equations of the form (3.12) and to obtain formulas for the generating functions of the solution of equation (3.11).

Let $x, m, \alpha \in \mathbb{Z}^N_{\geqslant}, P(z) = \sum\limits_{0\leqslant\alpha\leqslant m} c_\alpha z^\alpha$ be a polynomial in $z \in \mathbb{C}^N$. The inequality $0 \leqslant \alpha \leqslant m$ means that $0 \leqslant \alpha_j \leqslant m_j$ for all $j = 1, \dots, N$. We denote $F_\alpha(z) = \sum\limits_{x\geqslant\alpha} f(x)z^x$ and $\Phi_\alpha(z) = F(z) - F_\alpha(z)$, where the inequality $x \not\geqslant \alpha$ means, that for at least one $j_0 \in \{1, \dots, N\}$ the inequality $x_{j_0} < \alpha_{j_0}$ holds.

Let δ_j be a shift operator over j^{th} variable: $\delta_j f(x) = f(x_1, \dots, x_{j-1}, x_j + 1, x_{j+1}, \dots, x_N)$, then $\delta^\alpha = \delta_1^{\alpha_1} \circ \dots \circ \delta_n^{\alpha_n}$ and $P(\delta) = \sum\limits_{0\leqslant\alpha\leqslant m} c_\alpha \delta^\alpha$ be a polynomial difference operator with constant coefficients.

We first derive a general identity for the generating functions.

Theorem 3.1. *For any $F(z) \in \mathbb{C}[[z]]$ the identity*

$$P(z)F(z) - \sum_{0\leqslant\alpha\leqslant m} c_\alpha z^\alpha \Phi_{m-\alpha}(z) = \sum_{x\geqslant m} P(\delta^{-I}) f(x) z^x \tag{3.13}$$

holds, where $I = (1, \dots, 1)$.

Proof. Let $F(z) = \sum\limits_{x\geqslant 0} f(x) z^x$ and $P(z) = \sum\limits_{0\leqslant\alpha\leqslant m} c_\alpha z^\alpha$ and consider the product

$$P(z) \cdot F(z) = \Big(\sum_{0\leqslant\alpha\leqslant m} c_\alpha z^\alpha\Big)\Big(\sum_{x\geqslant 0} f(x) z^x\Big) =$$

$$= \sum_{0\leqslant\alpha\leqslant m} c_\alpha z^\alpha \Big(\sum_{x\geqslant m-\alpha} f(x) z^x + \sum_{x\ngeqslant m-\alpha} f(x) z^x\Big) =$$

$$= \sum_{0\leqslant\alpha\leqslant m} c_\alpha \sum_{x\geqslant m-\alpha} f(x) z^{x+\alpha} + \sum_{0\leqslant\alpha\leqslant m} c_\alpha z^\alpha \sum_{x\ngeqslant m-\alpha} f(x) z^x =$$

$$= \sum_{0\leqslant\alpha\leqslant m} c_\alpha \sum_{x\geqslant m} f(x-\alpha) z^x + \sum_{0\leqslant\alpha\leqslant m} c_\alpha z^\alpha \sum_{x\ngeqslant m-\alpha} f(x) z^x =$$

$$= \sum_{x\geqslant m}\Big(\sum_{0\leqslant\alpha\leqslant m} c_\alpha f(x-\alpha)\Big) z^x + \sum_{0\leqslant\alpha\leqslant m} c_\alpha z^\alpha \sum_{x\ngeqslant m-\alpha} f(x) z^x.$$

Since $\sum\limits_{0\leqslant\alpha\leqslant m} c_\alpha f(x-\alpha) = P(\delta^{-I}) f(x)$ and $\Phi_{m-\alpha}(z) = \sum\limits_{x\ngeqslant m-\alpha} f(x) z^x$ we get (3.13). □

The identity (3.13) implies that for any function of initial data $\varphi(x),\ x \ngeqslant m,\ x \geqslant 0$ and any function $g(x),\ x \geqslant m$, the equation $P(\delta^{-I}) f(x) = g(x)$ has a unique solution $f(x)$ satisfying initial data: $f(x) = \varphi(x),\ x \geqslant 0,\ x \ngeqslant m$ (see [8], [30], [29]). If $G(z) = \sum\limits_{x\geqslant m} g(x) z^x$, then identity (3.13) gives

$$F(z) = \frac{1}{P(z)} \sum_{0\leqslant\alpha\leqslant m} c_\alpha z^\alpha \Phi_{m-\alpha}(z) + \frac{G(z)}{P(z)}.$$

Let $\Delta = \{e^1, \ldots, e^N\}$, where the vector $e^j = (0, \ldots, 0, 1, 0, \ldots, 0)$ contains a unit on the j^{th} place for $j = 1, \ldots, N$, we denote $f(x)$ the number of paths from the origin to the point $x \in \mathbb{Z}^N_{\geqslant}$.

Corollary 3.2. *If $f(x)$ is the number of lattice paths from the origin to $x \in \mathbb{Z}^N_{\geqslant}$ using steps from the set Δ, then its generating function $F(z)$ equals to*

$$F(z) = \frac{1}{1 - z_1 - \ldots - z_N}.$$

Proof. We note that the function $f(x)$ satisfies to the basic recurrence relation $f(x) = f(x - e^1) + \cdots + f(x - e^N)$, which implies that the right side of the identity (3.13) equals 0.

Let us write the identity (3.13) for the two dimensional case:

$$(1 - z_1 - z_2)F(z_1, z_2) - (1 - z_2)F(0, z_2) - (1 - z_1)F(z_1, 0) + F(0, 0) = 0.$$

Since $f(x_1, 0) = f(0, x_2) = 1$ for all nonnegative integers x_1 and x_2, we obtain

$$F(0, z_2) = \frac{1}{1 - z_2}, \quad F(z_1, 0) = \frac{1}{1 - z_1}, \quad F(0, 0) = 1,$$

hence $F(z_1, z_2) = \frac{1}{1 - z_1 - z_2}$.

For $N = 3$ we have

$(1-z_1-z_2-z_3)F(z_1, z_2, z_3) - (1-z_1-z_2)F(z_1, z_2, 0) - (1-z_1-z_3)F(z_1, 0, z_3) - (1-z_2-z_3)F(0, z_2, z_3) + (1-z_1)F(z_1, 0, 0) + (1-z_2)F(0, z_2, 0) + (1-z_3)F(0, 0, z_3) - F(0, 0, 0) = 0.$

Considering three dimensional case, we obtain

$$F(z_1, z_2, z_3) = \frac{1}{1 - z_1 - z_2 - z_3}.$$

Repeating this process we get the generating function for any $N > 1$. □

We will demonstrate another way of using identity (3.13) for a two-dimensional case which is useful for some lattice path problems.

We denote $F(z_1,z_2)=\sum\limits_{(x_1,x_2)\geqslant(0,0)} f(x_1,x_2)z_1^{x_1}z_2^{x_2}$ and

$F_{p,q}(z_1,z_2)=\sum\limits_{k=1}^{\infty} f(pk,qk)z_1^{pk}z_2^{qk}$, where $(p,q)\in\mathbb{Z}^2_{\geqslant}$.

Let the function $f(x,y)=\varphi(x,y),\ (x,y)\ngeqslant(p,q),\ (x,y)\geqslant 0$ satisfy the difference equation

$$P(\delta_1^{-1},\delta_2^{-1})f(x,y)=g(x,y),$$

where $g(x,y)=\begin{cases} f(x,y), & \text{if}\quad x=pk,\ y=qk,\ k\geqslant 1\\ 0, & otherwise\end{cases}$, then

$F_{p,q}(z_1,z_2)=\sum\limits_{(x,y)\geqslant(p,q)} g(x,y)z_1^x z_2^y$ and identity (3.13) is

$$P(z_1,z_2)F(z_1,z_2)-\sum_{\substack{0\leqslant\alpha_1\leqslant p\\0\leqslant\alpha_2\leqslant q}} c_{\alpha_1,\alpha_2}z_1^{\alpha_1}z_2^{\alpha_2}\Phi_{p-\alpha_1,q-\alpha_2}(z_1,z_2)=F_{p,q}(z_1,z_2).$$

If $z_1=z_1(t),\ z_2=z_2(t)$ is a solution to a system $\begin{cases} z_1^p z_2^q=t\\ P(z_1,z_2)=0\end{cases}$, then function $F_{p,q}$ satisfies the formulae

$$F_{p,q}(z_1(t),z_2(t))=-\sum_{\substack{0\leqslant\alpha_1\leqslant p\\0\leqslant\alpha_2\leqslant q}} c_{\alpha_1,\alpha_2}z_1^{\alpha_1}(t)z_2^{\alpha_2}(t)\Phi_{p-\alpha_1,q-\alpha_2}\big(z_1(t),z_2(t)\big).$$

Now we will present the examples connected with some well-known lattice paths: Dyck, Motzkin and Schröder paths (see [8], [27], [40], [18], [13], [12], [55]). We use a linear transformation which transforms the mentioned lattice paths to lattice paths lying in $\mathbb{Z}^2_{\geqslant}$ in order to use methods for finding generating functions developed [30] and [26]. However, to study lattice paths lying on or over a line having rational slope, linear difference equations with non-constant coefficients will be used to incorporate this restriction.

3.3 Dyck, Schröder, Motzkin paths

In this paragraph we will illustrate methods developed in paragraph 2.2 and consider Dyck paths, Schröder paths, Motzkin paths, and other lattice paths.

Dyck paths start at the origin and stay on or above the main diagonal $y = x$ (see [8], [40], [41]) using steps $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Let $f(x, y)$ denote the number of paths going from $(0, 0)$ to (x, y). The number of paths $f(x, y)$ satisfies the difference equation

$$f(x, y) - f(x-1, y) - f(x, y-1) = -\delta_0(x-y-1)f(x-1, y), \qquad (3.14)$$

where $\delta_0(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x \neq 0 \end{cases}$ is a Kronecker symbol, with the initial data:

$$f(x, 0) = 0, \;\; x = 1, 2, \dots, \quad f(0, y) = 1, \;\; y = 0, 1, 2, \dots. \qquad (3.15)$$

Let $F_{11}(t)$ be the diagonal power series of $F(z_1, z_2)$:

$$F_{11}(t) = \sum_{k=1}^{\infty} f(k, k)t^k.$$

Proposition 3.3. *Let $F(z_1, z_2)$ be the generating function of the solution of (3.14). Then the series $F(z_1, z_2)$ satisfy the following functional equation*

$$(1 - z_1 - z_2)F(z_1, z_2) - (1 - z_1)F(z_1, 0) - (1 - z_2)F(0, z_2) + F(0, 0) =$$
$$= -z_1 \sum_{k \geqslant 1} f(k, k)(z_1 z_2)^k.$$

If the solution of $f(x, y)$ satisfies the initial conditions (3.15), then we get a diagonal power series

$$F_{11}(t) = \sum_{k=1}^{\infty} f(k, k)t^k = \frac{1 - 2t - \sqrt{1 - 4t}}{2t} = t + 2t^2 + 5t^3 + 14t^4 + 42t^5 \cdots . \qquad (3.16)$$

Proof. For $N = 2$ and $P(z_1, z_2) = 1 - z_1 - z_2$ we have $c_{00} = 1,\ c_{10} = c_{01} = -1,\ c_{11} = 0,\ m = (1, 1),\ \Phi_{1,1}(z_1, z_2) = F(z_1, 0) + F(0, z_2) - F(0, 0),\ \Phi_{1,0}(z_1, z_2) = F(0, z_2),\ \Phi_{0,1}(z_1, z_2) = F(z_1, 0),\ \Phi_{0,0}(z_1, z_2) = 0$.

Then by theorem 3.1 we get

$$(1 - z_1 - z_2)F(z_1, z_2) - c_{00}\Phi_{1,1}(z_1, z_2) - c_{10}z_1\Phi_{0,1}(z_1, z_2) - c_{01}z_2\Phi_{1,0}(z_1, z_2) = \\ = \sum_{\substack{x\geqslant 1\\ y\geqslant 1}} (1 - \delta_1^{-I} - \delta_2^{-I}) f(x, y) z_1^x z_2^y.$$

Using the difference equation (3.14) implies

$$(1 - z_1 - z_2)F(z_1, z_2) - (1 - z_1)F(z_1, 0) - (1 - z_2)F(0, z_2) + F(0, 0) = \\ = -z_1 \sum_{k\geqslant 1} f(k, k)(z_1 z_2)^k,$$

where $F(0, z_2) = \sum\limits_{y\geqslant 0} f(0, y) z_2^y = \frac{1}{1-z_2},\ F(z_1, 0) = \sum\limits_{x\geqslant 0} f(x, 0) z_1^x = 1,\ F(0, 0) = f(0, 0) = 1$.

Let $P(z_1, z_2) = 1 - z_1 - z_2 = 0$, we obtain

$$-1 + \frac{1}{z_1} = \sum_{k=1}^{\infty} f(k, k)(z_1(1 - z_1))^k.$$

The substitution $t = z_1(1 - z_1)$ yields $z_1 = \frac{1+\sqrt{1-4t}}{2}$ and after expansion $\frac{1}{z_1}$ in a series, we obtain

$$\sum_{k=1}^{\infty} f(k, k)t^k = -1 + \frac{1 - \sqrt{1 - 4t}}{2t} = t + 2t^2 + 5t^3 + 14t^4 + 42t^5 \cdots ,$$

which proves the proposition. □

The coefficients of the series (3.16) represent the Catalan numbers $f(k, k) = \frac{1}{k-1}\begin{pmatrix} 2k \\ k \end{pmatrix}, k \geqslant 1$.

Schröder paths start at the origin and stay on or above the main diagonal $y = x$ (see [8]) using steps $(1, 0), (0, 1), (1, 1)$. Let $f(x, y)$ denote the number of paths going from $(0,0)$ to (x,y). The number of paths $f(x,y)$ satisfies the difference equation

$$f(x,y)-f(x-1,y)-f(x,y-1)-f(x-1,y-1) = -\delta_0(x-y-1)f(x-1,y), \quad (3.17)$$

with the initial data:

$$f(x,0)=0,\ \ x=1,2,\dots, \quad f(0,y)=1,\ \ y=0,1,2,\dots. \qquad (3.18)$$

Proposition 3.4. *Let $F(z_1,z_2)$ be the generating function of the solution of (3.17). Then the series $F(z_1,z_2)$ satisfy the following functional equation*

$$(1-z_1-z_2-z_1z_2)F(z_1,z_2)-(1-z_1)F(z_1,0)-(1-z_2)F(0,z_2)+F(0,0)= \\ = -z_1\sum_{k\geqslant 1} f(k,k)(z_1z_2)^k.$$

If the solution of $f(x,y)$ satisfies the initial conditions (3.18), then we obtain a diagonal power series

$$F_{11}(t)=\sum_{k=1}^{\infty} f(k,k)t^k = \frac{1-3t-\sqrt{1-6t+t^2}}{2t} = 2t+6t^2+22t^3+90t^4+\cdots. \qquad (3.19)$$

Proof. Using theorem 3.1 and difference equation (3.17) implies

$$(1-z_1-z_2-z_1z_2)F(z_1,z_2)-(1-z_1)F(z_1,0)-(1-z_2)F(0,z_2)+F(0,0)= \\ = -z_1\sum_{k\geqslant 1} f(k,k)(z_1z_2)^k,$$

where $F(0,z_2)=\sum\limits_{y\geqslant 0} f(0,y)z_2^y = \frac{1}{1-z_2}$, $F(z_1,0)=\sum\limits_{x\geqslant 0} f(x,0)z_1^x = 1$, $F(0,0)= f(0,0)=1$.

Let $P(z_1,z_2)=1-z_1-z_2-z_1z_2=0$, we obtain

$$-1+\frac{1}{z_1}=\sum_{k=1}^{\infty} f(k,k)\left(\frac{z_1(1-z_1)}{1+z_1}\right)^k.$$

The substitution $t=\frac{z_1(1-z_1)}{1+z_1}$ yields $z_1=\frac{1-t+\sqrt{1-6t+t^2}}{2}$ and after expansion $\frac{1}{z_1}$ in a series, we obtain (3.19). □

The coefficients of the series (3.19) coincides with the numbers of the Schröder paths ending on the main diagonal $y = x$.

Motzkin paths start at the origin and stay on or above the main diagonal $y = x$ (see [13]) using steps $(2, 0), (0, 2), (1, 1)$. Let $f(x, y)$ denote the number of paths going from $(0, 0)$ to (x, y). The number of paths $f(x, y)$ satisfies the difference equation

$$f(x,y) - f(x-2,y) - f(x,y-2) - f(x-1,y-1) = \\ = -(\delta_0(x-y-1) + \delta_0(x-y-2))f(x-2,y), \quad (3.20)$$

with the initial data:

$$f(x,0) = 0,\ \ x = 1,2,3,\dots, \quad f(0,y) = \frac{1+(-1)^y}{2},\ \ y = 0,1,2,\dots, \\ f(x,1) = 0,\ \ x = 2,3,4,\dots, \quad f(1,y) = \frac{(1-(-1)^y)(y+1)}{4},\ \ y = 1,2,3,\dots. \quad (3.21)$$

Proposition 3.5. *Let $F(z_1, z_2)$ be the generating function of the solution of (3.20). Then the series $F(z_1, z_2)$ satisfy the following functional equation*

$$(1 - z_1^2 - z_2^2 - z_1z_2)F(z_1,z_2) - (1-z_1)F(z_1,0) - (1-z_2)F(0,z_2) + F(0,0) = \\ = -z_1^2\sum_{k\geqslant 2} f(k,k)(z_1z_2)^k.$$

If the solution of $f(x, y)$ satisfies the initial conditions (3.21), then we obtain a diagonal power series

$$F_{11}(t) = \sum_{k=2}^{\infty} f(k,k)t^k = -1-t+\frac{1-t-\sqrt{1-2t-3t^2}}{2t^2} = 2t^2+4t^3+9t^4+21t^5+\cdots. \quad (3.22)$$

Proof. Using theorem 3.1 and difference equation (3.20) implies

$$(1-z_1^2-z_2^2-z_1z_2)F(z_1,z_2) - (1-z_1^2-z_1z_2)F(z_1,0) - (1-z_2^2-z_1z_2)F(0,z_2)+ \\ + F(0,0)(1-z_1z_2) - \tilde{\Phi}_{1,0}(z_1,z_2)(1-z_2^2) - \tilde{\Phi}_{0,1}(z_1,z_2)(1-z_1^2) + f(1,1)z_1z_2 = \\ = -z_1^2\sum_{k\geqslant 2} f(k,k)(z_1z_2)^k,$$

where $F(z_1,0)=1,\ F(0,z_2)=\frac{1}{1-z_2^2},\ F(0,0)=1,\ \tilde{\Phi}_{0,1}(z_1,z_2)=z_1z_2,\ \tilde{\Phi}_{1,0}(z_1,z_2)=\frac{z_1z_2}{(1-z_2^2)^2},\ f(1,1)=1.$

Let $P(z_1,z_2)=1-z_1^2-z_2^2-z_1z_2=0$, we obtain

$$-z_1z_2-1+\frac{1}{z_1^2}=\sum_{k=2}^{\infty}f(k,k)\left(\frac{z_1(\sqrt{4-3z_1^2}-z_1)}{2}\right)^k.$$

The substitution $t=z_1z_2$ yields $z_1^2=\frac{1-t+\sqrt{1-2t-3t^2}}{2}$ and after expansion $\frac{1}{z_1^2}$ in a series, we obtain (3.22). □

The coefficients of the series (3.22) coincides with the numbers of the Motzkin paths ending on the main diagonal $y=x$.

Consider generalizing the problem of enumerating lattice paths with steps $(1,0),(0,1),(r$ which start at the origin and stay on or above the main diagonal $y=x$. Let $f(x,y)$ denote the number of paths going from $(0,0)$ to (x,y). The number of paths $f(x,y)$ satisfies the difference equation

$$f(x,y)-f(x-1,y)-f(x,y-1)-f(x-r,y-r)=-\delta_0(x-y-1)f(x-1,y), \quad (3.23)$$

with some defined initial data :

$$f(x,y)=\varphi(x,y),(x,y)\geqslant(0,0),(x,y)\ngeqslant(r,r). \quad (3.24)$$

Proposition 3.6. *Let $F(z_1,z_2)$ be the generating function of the solution of (3.23). Then the series $F(z_1,z_2)$ satisfy the following functional equation*

$$(1-z_1-z_2-z_1^rz_2^r)F(z_1,z_2)-\Phi_{r,r}(z_1,z_2)+z_1\Phi_{r-1,r}(z_1,z_2)+z_2\Phi_{r,r-1}(z_1,z_2)=$$
$$=-z_1\sum_{k\geqslant r}f(k,k)(z_1z_2)^k.$$

Proof. Using theorem 3.1 and difference equation (3.23) implies

$$(1-z_1-z_2-z_1^rz_2^r)F(z_1,z_2)-\Phi_{r,r}(z_1,z_2)+z_1\Phi_{r-1,r}(z_1,z_2)+z_2\Phi_{r,r-1}(z_1,z_2)=$$
$$=-z_1\sum_{k\geqslant r}f(k,k)(z_1z_2)^k,$$

where

$$\Phi_{r,r}(z_1,z_2)=\Phi_{r-1,r-1}(z_1,z_2)+\tilde\Phi_{0,r-1}(z_1,z_2)+\tilde\Phi_{r-1,0}(z_1,z_2)-f(r-1,r-1)(z_1z_2)^{r-1},$$

$$\Phi_{r-1,r}(z_1,z_2)=\Phi_{r-1,r-1}(z_1,z_2)+\tilde\Phi_{0,r-1}(z_1,z_2),$$

$$\Phi_{r,r-1}(z_1,z_2)=\Phi_{r-1,r-1}(z_1,z_2)+\tilde\Phi_{r-1,0}(z_1,z_2),\ \textit{and}$$

$$\Phi_{r-1,r-1}(z_1,z_2)=\sum_{i=0}^{r-2}\tilde\Phi_{i,0}(z_1,z_2),\tilde\Phi_{0,r-1}(z_1,z_2)=\sum_{x=r-1}^{\infty}f(x,r-1)z_1^xz_2^{r-1},$$

$$\tilde\Phi_{r-1,0}(z_1,z_2)=\sum_{y=r-1}^{\infty}f(r-1,y)z_1^{r-1}z_2^y.$$

Let $P(z_1,z_2)=1-z_1-z_2-z_1^rz_2^r=0$, we obtain

$$\Phi_{r-1,r-1}(z_1,z_2)+\tilde\Phi_{0,r-1}(z_1,z_2)+\tilde\Phi_{r-1,0}(z_1,z_2)-f(r-1,r-1)(z_1z_2)^{r-1}-$$

$$-z_1(\Phi_{r-1,r-1}(z_1,z_2)+\tilde\Phi_{0,r-1}(z_1,z_2))-z_2(\Phi_{r-1,r-1}(z_1,z_2)+\tilde\Phi_{r-1,0}(z_1,z_2))=$$

$$=z_1\sum_{k\geqslant r}f(k,k)(z_1z_2)^k.$$

Since $f(x,y)=0$ below the diagonal, we obtain

$$(1-z_1-z_2)\Phi_{r-1,r-1}(z_1,z_2)+(1-z_2)\tilde\Phi_{r-1,0}(z_1,z_2)=z_1\sum_{k\geqslant r-1}f(k,k)(z_1z_2)^k.$$

The substitution $t=z_1z_2$ implies $z_1=\frac{1-t^r+\sqrt{1-4t-2t^r+t^{2r}}}{2}$ and then

$$\frac{1}{z_1}((1-z_1-z_2)\Phi_{r-1,r-1}(z_1,z_2)+(1-z_2)\tilde\Phi_{r-1,0}(z_1,z_2))=\sum_{k\geqslant r-1}f(k,k)t^k. \quad (3.25)$$

For $r=2$ we have the initial data: $f(x,0)=0,\ \ x=1,2,3,\dots,\ \ f(0,y)=1,\ \ y=0,1,2,\dots,\ \ f(x,1)=0,\ \ x=2,3,4,\dots,\ \ f(1,y)=y,\ \ y=1,2,3,\dots$.

Using (3.25) implies

$$-1+\frac{1}{z_1}=\sum_{k=1}^{\infty}f(k,k)(z_1z_2)^k.$$

The substitution $t = z_1 z_2$ implies $z_1 = \frac{1-t^2+\sqrt{1-4t-2t^2+t^4}}{2}$ and after expansion $\frac{1}{z_1}$ in a series, we obtain

$$-1+\frac{1}{z_1} = -1+\frac{1-t^2-\sqrt{1-4t-2t^2+t^4}}{2t} = t+3t^2+8t^3+25t^4+83t^5+\cdots\,, \tag{3.26}$$

which proves the proposition. □

The coefficients of the series (3.26) coincide with the numbers of other paths ending on the main diagonal $y = x$.

CONCLUSION

The main results of the thesis

- we obtained formulae in which the generating function of the solution to the Cauchy problem is expressed in terms of generating functions of the Cauchy data and a solution to the Cauchy problem is expressed through its fundamental solution and Cauchy data;
- we obtained the Chaundy-Bullard identity for vector partition functions;
- we obtained the identity for generating functions (series), based on which we derived generating functions of solutions to restricted lattice path problems.

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